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MECHANICS OF A NEAR NET-SHAPE STRESS-COATED MEMBRANE

Volume I of II
THEORY DEVELOPMENT USING THE METHOD OF ASYMPTOTIC EXPANSIONS

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14. ABSTRACT

The method of asymptotic expansions was applied to the geometrically nonlinear, three-dimensional equations of a coated membrane laminate, each material component of which was assumed to be a linear, uniform, homogeneous, and isotropic elastic material in which there exist residual stresses. Our goal was to systematically derive by a single method the generalizations of four well-known theories of a single material to a coated membrane laminate. Two of the theories, one geometrically linear, the other geometrically nonlinear, describe a true membrane laminate offering no resistance to bending. These are applicable to membrane laminate vibration analysis, and pressurized stress-coated membranes undergoing large deflections, respectively. The other two describe stress-coated membrane shells having small but non-zero bending stiffness; these theories are to be used to determine a coating stress prescription that will maintain the shape of an initially parabolic stress-coated membrane laminate. Solutions of associated boundary value problems are given in Volume II of the report.

15. SUBJECT TERMS
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1 Introduction

The primary motivation for this research is the interest by the Air Force and many other organizations in developing and deploying large, precise, lightweight, space-based antennas and optical telescopes. Large diameter, optical quality membrane reflectors may well be the critical components that make such structures possible. The prevailing paradigm for creating doubly-curved membrane surfaces is the pressurized lenticular configuration, illustrated in Figure 1. However, there are serious difficulties that must be overcome in order for such a configuration to be successfully deployed in space. These problems are discussed at some length in Reference [1].

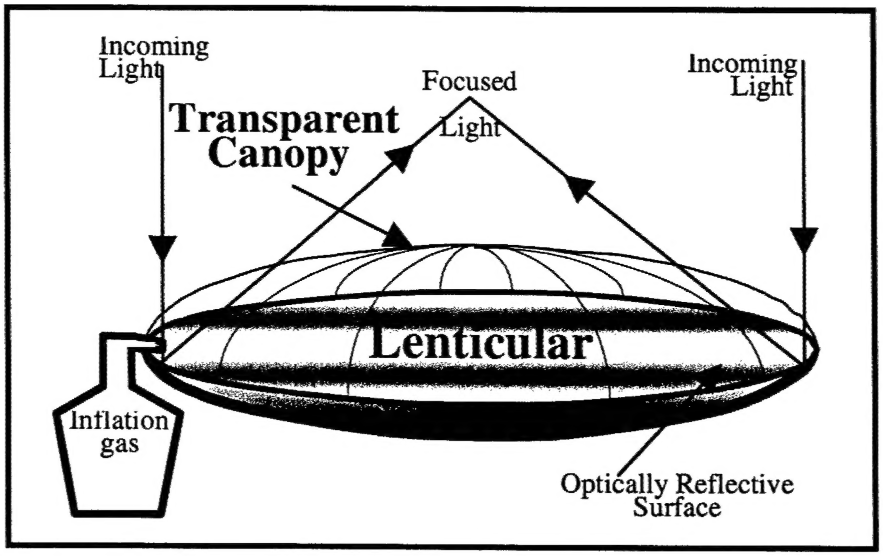

Figure 1: Optical lenticular for imaging.

An alternative to the pressurized lenticular configuration is a thin laminate shell consisting of a membrane and a dielectric coating commonly used in optics to meet a high reflectivity requirement. This laminate shell is referred to as a stress-coated net-shape membrane reflector. The net-shape membrane portion of the laminate is realized by manufacturing a polymer to nearly its final shape, which is typically either spherical or parabolic. The net-shape process involves several basic steps. The solvent-based polymer is initially cast on a mandrel (also referred to as a "mold") having the desired surface qualities (e.g., optically smooth, and having the required shape), where the solvent is allowed to evaporate, leaving a thin high quality membrane of the desired shape. At this point, however, significant internal stress has developed in the membrane due to the solvent evaporation process. The membrane is then taken through an annealing process in which the mold and membrane are heated to nearly the glass transition temperature of the polymer, eliminating most of the shrinkage stress. However, the coefficient of thermal expansion (CTE) of the membrane is, in the systems we consider, higher than that of the mold. As the system is allowed to cool to room temperature, the membrane attempts to contract more than the mold, due to its higher CTE. Since the membrane is fully constrained by the mold, hence cannot complete its contraction, the effect of the CTE mismatch is to induce a (tensile) thermal stress in the membrane. Such a stress is referred to as *non-mechanical* (or *inelastic* or *residual*), i.e., it is a stress that exists in the absence of any displacement-related strain (see, for example, Fung [2], pp. 354–355). This residual thermal stress would act to deform the membrane from its initial shape upon removal from the mold. The other serious problem is the simple fact that a membrane is "flimsy", that is, it lacks the stiffness required to resist bending due to external loads. Examples of such loads are gravity

and wind in a near earth environment, and slewing or other forces used for control in a space environment. At any rate, upon removal from the mold, a net-shape membrane would not be expected to retain the shape of the mold.

The research reported here addresses the possibility of solving both the CTE mismatch problem, and lack of stiffness, by applying to the membrane a coating with an intrinsic compressive stress designed to compensate the CTE mismatch stress, as well as provide enough stiffness to maintain the desired shape under various loads. Specifically, we examine the effects of gravity and uniform pressure loads on the stress-coated membrane. In Volume I of this two-volume report the method of asymptotic expansions is used to derive various theories of stress-coated membranes from the general, geometrically nonlinear, three-dimensional theory of elasticity. In Volume II we present solutions of the equations satisfying various types of boundary conditions.

2 Reference Placement and Reference Configuration

We introduce a region $\mathcal{C}$ of 3-dimensional Euclidean space in the form of a thin right circular cylinder of radius a and uniform thickness (or height) $h << a$, and refer to this purely mathematical construct as the *reference placement* of a coated membrane shell. This cylinder is further assumed to be divided into two coaxial cylinders of the same radius a, one of thickness h_s, the other of thickness h_c, so that $h = h_s + h_c$, as shown in the lower portions of Figures 2 and 3. We assume given a fixed orthonormal Cartesian basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ with origin $\mathbf{O}$ at the center of the circular disk defined by the intersection of $\mathcal{C}$ and a bisecting plane orthogonal to the axis. An arbitrary point $\mathbf{P}$ of the reference placement may be specified by either its Cartesian coordinates $X^A = \{X, Y, Z\}$, or its cylindrical coordinates $Q^A = \{R, \Theta, Z\}$. Thus, the bisecting or middle plane of $\mathcal{C}$ is defined by $Z = 0$, and the axis of $\mathcal{C}$ by the line $X = Y = 0$ through $\mathbf{O}$. The position vector of $\mathbf{P}$ with respect to $\mathbf{O}$ is given by

$$\mathbf{X} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} = R\cos\Theta\,\mathbf{i} + R\sin\Theta\,\mathbf{j} + Z\mathbf{k}. \tag{2.1}$$

We introduce orthonormal basis vectors $\{\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z\}$ associated with the cylindrical coordinates, defined by

$$\mathbf{E}_A \equiv \mathbf{X}_{,A}/|\mathbf{X}_{,A}|, \qquad \text{where} \qquad \mathbf{X}_{,A} \equiv \frac{\partial \mathbf{X}}{\partial Q^A}, \tag{2.2}$$

so that

$$\mathbf{E}_R = \cos\Theta\,\mathbf{i} + \sin\Theta\,\mathbf{j}, \quad \mathbf{E}_\Theta = -\sin\Theta\,\mathbf{i} + \cos\Theta\,\mathbf{j}, \quad \mathbf{E}_Z = \mathbf{k}, \tag{2.3}$$

in terms of which we can write the position vector as

$$\mathbf{X} = R\mathbf{E}_R + Z\mathbf{E}_Z. \tag{2.4}$$

The *physical* system of interest is a laminate material body in the form of an initially curved membrane substrate to which an optical coating has been applied. The *reference configuration* of this coated membrane is assumed to be a region $\mathcal{S}$ defined by a mapping ϕ from the reference placement $\mathcal{C}$, under which a point $\mathbf{P}$ of $\mathcal{C}$ is mapped to some material point $\widetilde{\mathbf{P}} = \phi(\mathbf{P})$ of $\mathcal{S}$. A point $\mathbf{S}$ of the middle plane of $\mathcal{C}$ with coordinates $(R, \Theta, 0)$ is mapped by ϕ to a point $\widetilde{\mathbf{S}}$ of the middle *surface* of $\mathcal{S}$ with coordinates $(R, \Theta, \widetilde{Z}_S)$, where

$$\widetilde{Z}_S = \Gamma(R), \tag{2.5}$$

hence we are assuming that the middle surface is a surface of revolution. The azimuthal coordinate Θ of any point of $\mathcal{C}$ is assumed unchanged by this mapping, so that $\widetilde{\Theta} = \Theta$ on $\mathcal{S}$. This action of ϕ on the middle plane is illustrated in the upper portions of Figures 2 and 3.

The action of ϕ on points *off* the middle plane depends on the distribution of the thicknesses h_c and h_s as a result of the processes used to cast the membrane on the mold, and to apply the coating to the membrane. The simplest model results by assuming the coated membrane to have constant *axial* thicknesses, so that

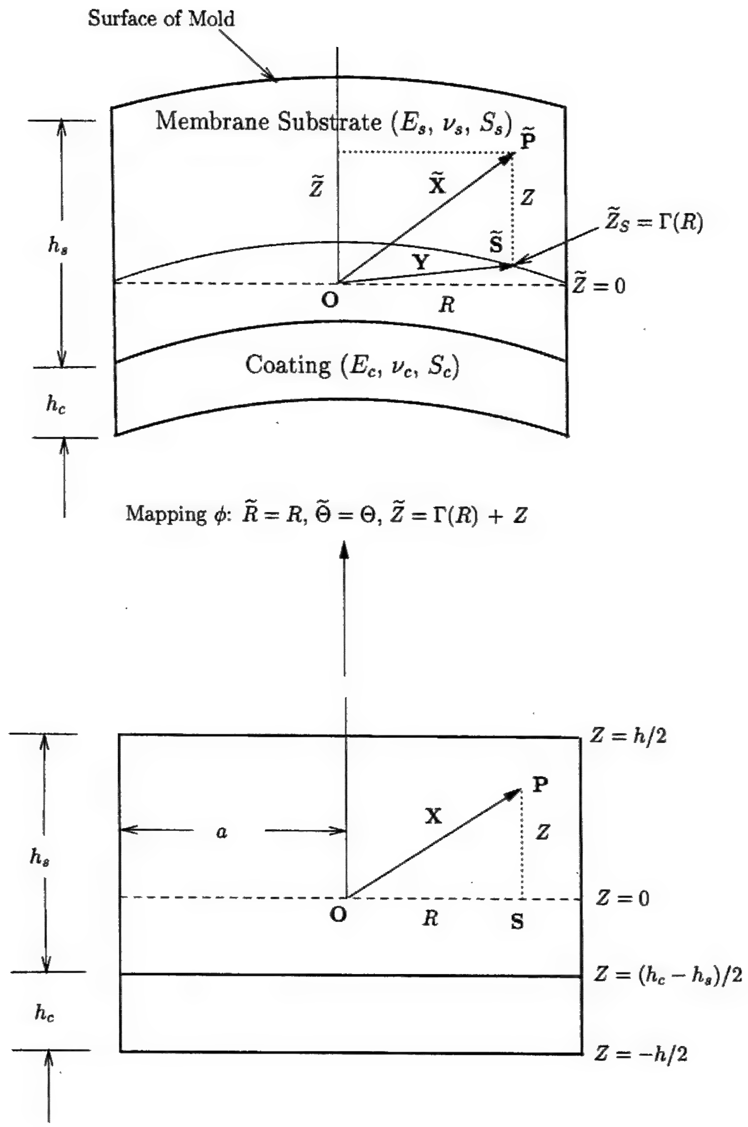

Figure 2: Definition of the reference configuration $\mathcal{S}$ (upper part of Figure) of a coated membrane shell of revolution as a mapping from the reference placement $\mathcal{C}$ (lower part of Figure), assuming the thicknesses h_c and h_s to be constant along any line parallel to the axis.

an arbitrary point $\mathbf{P}$ with non-zero axial coordinate Z is mapped to a material point $\widetilde{\mathbf{P}}$ with coordinates $\{\widetilde{R}, \Theta, \widetilde{Z}\}$, obtained by translating a distance Z (equal to the original axial coordinate) from $\widetilde{\mathbf{S}}$ along the axial direction $\mathbf{E}_Z$, as shown in the upper portion of Figure 2. The complete mapping can be determined geometrically as follows. The position vector of $\widetilde{\mathbf{P}}$ with respect to $\mathbf{O}$ can, according to Figure 2, be written in two different ways:

$$\widetilde{\mathbf{X}} = \widetilde{R}\,\mathbf{E}_R + \widetilde{Z}\,\mathbf{E}_Z = \mathbf{Y} + Z\,\mathbf{E}_Z, \tag{2.6}$$

where $\mathbf{Y}$ is the position vector of $\widetilde{\mathbf{S}}$ with respect to $\mathbf{O}$. Now, $\mathbf{Y} = R\,\mathbf{E}_R + \Gamma(R)\,\mathbf{E}_Z$, hence we have from

(2.6):

$$\widetilde{R}\,\mathbf{E}_R + \widetilde{Z}\,\mathbf{E}_Z = R\,\mathbf{E}_R + [\Gamma(R) + Z]\,\mathbf{E}_Z,$$

and a comparison of components on both sides of the second equality yields the remaining two component mappings of ϕ ($\widetilde{\Theta} = \Theta$ is the other component mapping):

$$\widetilde{R} = R, \quad \text{and} \quad \widetilde{Z} = \Gamma(R) + Z. \tag{2.7}$$

The position vector of a material point in the reference configuration $\mathcal{S}$ of the coated membrane is thus given in terms of coordinates on the reference placement $\mathcal{C}$ by

$$\widetilde{\mathbf{X}} = R\,\mathbf{E}_R + [\Gamma(R) + Z]\,\mathbf{E}_Z. \tag{2.8}$$

For comparisons with finite element analyses that use shell elements in their formulation, it is perhaps more appropriate to assume the coated membrane to have constant thicknesses h_c and h_s *normal* to the middle surface. In this case, an arbitrary point $\mathbf{P}$ with non-zero axial coordinate Z is mapped to a material point $\widetilde{\mathbf{P}}$ with coordinates $\{\widetilde{R}, \Theta, \widetilde{Z}\}$, obtained by translating the distance Z along the *unit normal* $\mathbf{E}_N$ to the middle surface at $\widetilde{\mathbf{S}}$ (see Figure 3). The position vector of $\widetilde{\mathbf{P}}$ with respect to $\mathbf{O}$ is then given by

$$\widetilde{\mathbf{X}} = \widetilde{R}\,\mathbf{E}_R + \widetilde{Z}\,\mathbf{E}_Z = \mathbf{Y} + Z\,\mathbf{E}_N, \tag{2.9}$$

where $\mathbf{Y} = R\,\mathbf{E}_R + \Gamma(R)\,\mathbf{E}_Z$, as before. To compute the unit normal $\mathbf{E}_N$, we first note that equation (2.5) can be written as $\psi(R, \widetilde{Z}_S) = 0$, where ψ is the function defined by

$$\psi(R, \widetilde{Z}) \equiv \widetilde{Z} - \Gamma(R). \tag{2.10}$$

The unit normal to the midsurface is the normalized gradient of this function:

$$\mathbf{E}_N = \frac{\nabla\psi}{|\nabla\psi|} = \frac{-\Gamma_{,R}\,\mathbf{E}_R + \mathbf{E}_Z}{\sqrt{1 + (\Gamma_{,R})^2}}, \tag{2.11}$$

where $\Gamma_{,R}$ here is the *ordinary* derivative of Γ with respect to R (note that this slope is *negative* in the first quadrant). Substituting these results in (2.9) yields

$$\widetilde{R}\,\mathbf{E}_R + \widetilde{Z}\,\mathbf{E}_Z = R\,\mathbf{E}_R + \Gamma(R)\,\mathbf{E}_Z + Z\left(\frac{-\Gamma_{,R}\,\mathbf{E}_R + \mathbf{E}_Z}{\sqrt{1 + (\Gamma_{,R})^2}}\right),$$

from which the component mappings of ϕ are given by

$$\begin{aligned} \widetilde{R} &= R - Z\frac{\Gamma_{,R}}{\sqrt{1 + (\Gamma_{,R})^2}} = R + Z\sin\alpha, \\ \widetilde{Z} &= \Gamma(R) + Z\frac{1}{\sqrt{1 + (\Gamma_{,R})^2}} = \Gamma(R) + Z\cos\alpha, \end{aligned} \tag{2.12}$$

where α is the angle between $\mathbf{E}_Z$ and the unit normal $\mathbf{E}_N$ at a point on the middle surface ($\cos\alpha = \mathbf{E}_Z\cdot\mathbf{E}_N$). From (2.12) we have

$$\cos\alpha = \frac{1}{\Phi}, \qquad \sin\alpha = -\frac{\Gamma_{,R}}{\Phi}, \quad \text{where} \quad \Phi = \sqrt{1 + (\Gamma_{,R})^2}\,. \tag{2.13}$$

The position vector of a material point in the reference configuration of the coated membrane is thus given in terms of coordinates on the reference placement by

$$\widetilde{\mathbf{X}} = \left(R - Z\frac{\Gamma_{,R}}{\Phi}\right)\mathbf{E}_R + \left(\Gamma(R) + Z\frac{1}{\Phi}\right)\mathbf{E}_Z = (R + Z\sin\alpha)\,\mathbf{E}_R + [\Gamma(R) + Z\cos\alpha]\,\mathbf{E}_Z. \tag{2.14}$$

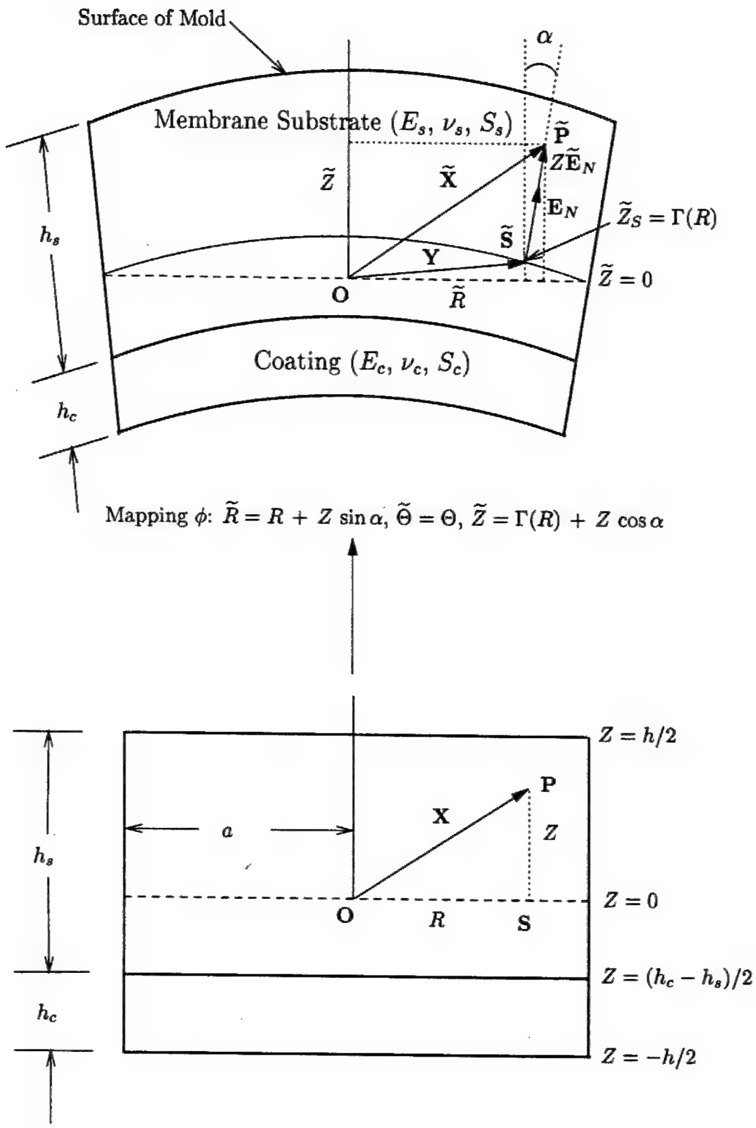

Figure 3: Definition of the reference configuration $\mathcal{S}$ as a mapping from the reference placement $\mathcal{C}$, assuming the thicknesses h_c and h_s to be constant along any line through a normal to the middle surface $\widetilde{Z}_S = \Gamma(R)$.

The use of a model with constant thicknesses normal to the middle surface is considerably more complicated to analyze theoretically than one with constant axial thicknesses. In the remainder of this work we use the simpler model. To get an estimate of the errors made in choosing the constant-axial-thickness model over the constant-normal-thickness model, we consider the important case of a paraboloidal middle surface defined by

$$\Gamma(R) = \Gamma_0 - \frac{1}{4f} R^2, \tag{2.15}$$

where f is the focal length and Γ_0 is the apex displacement. Since $\Gamma(a) = 0$, it follows that $\Gamma_0 = a^2/(4f)$,

hence

$$\Gamma(R) = \frac{1}{4f}\left(a^2 - R^2\right). \tag{2.16}$$

The tangent of the slope angle is, from (2.13), $\tan\alpha = -\Gamma_{,R}$, so that

$$\tan\alpha = \frac{1}{2f}R. \tag{2.17}$$

The f-number of the paraboloid, which we denote by $F\#$, is defined by

$$F\# = \frac{f}{2a}, \tag{2.18}$$

so we can write the last two equations in terms of f-number as

$$\Gamma(R) = \frac{1}{8aF\#}\left(a^2 - R^2\right), \qquad \tan\alpha = -\Gamma_{,R} = \frac{1}{4aF\#}R. \tag{2.19}$$

The angle α has its maximum value at the edge $R = a$, in which case

$$\tan\alpha_{max} = \frac{1}{4F\#}. \tag{2.20}$$

For the optical applications envisioned here we expect to have f-numbers of 2 or greater, hence

$$\begin{aligned} \tan\alpha_{max} &\leq 0.125 \quad \Rightarrow \quad \alpha_{max} \leq 7.1^\circ, \\ \cos\alpha_{max} &\geq 0.992, \qquad \sin\alpha_{max} \leq 0.124. \end{aligned} \tag{2.21}$$

Use of the constant-axial-thickness model entails approximating $\cos\alpha \approx 1$ and $\sin\alpha \approx 0$, so for this lowest f-number one might expect any differences between our theoretical results, and finite element analysis results, to be at least partly attributable to these approximations.

3 Deformation, Displacement, and Strain

When the coated membrane shell is removed from the mold, releasing it from its constraints, it deforms until a new equilibrium configuration, which we refer to as the deformed, or current, configuration, is attained. The deformation is assumed to be described mathematically by a one-to-one invertible mapping $\tilde{f}$ that maps the body point located at $\tilde{\mathbf{P}}$ of the reference configuration $\mathcal{C}$ to a new point $\mathbf{p} \equiv \tilde{f}(\tilde{\mathbf{P}})$. The set of image points of $\tilde{f}$ defines the deformed configuration $\mathcal{D}$. Variables that refer to points of this new configuration will be denoted by lower case Latin letters, e.g., $x^a = \{x^1, x^2, x^3\} = \{x, y, z\}$ are Cartesian coordinates, and $q^a = \{q^1, q^2, q^3\} = \{r, \theta, z\}$ are cylindrical coordinates on the deformed configuration. Thus, in terms of these cylindrical coordinates, the mapping $\mathbf{p} = \tilde{f}(\tilde{\mathbf{P}})$ is coordinatized by

$$q^a(\mathbf{p}) = q^a(\tilde{f}(\tilde{\mathbf{P}})) = (q^a \circ \tilde{f})(\tilde{\mathbf{P}}) \equiv \tilde{f}^a(\tilde{\mathbf{P}}), \tag{3.1}$$

where the functional compositions $q^a \circ \tilde{f} \equiv \tilde{f}^a$ define the cylindrical component mappings of the mapping $\tilde{f}$. Assuming the arbitrary point $\tilde{\mathbf{P}}$ to be coordinatized by the cylindrical coordinates $\tilde{Q}^A \equiv \{\tilde{R}, \tilde{\Theta}, \tilde{Z}\}$ on the reference configuration, equation (3.1) can be written as

$$q^a(\mathbf{p}) = \tilde{f}^a[\tilde{Q}^1(\tilde{\mathbf{P}}), \tilde{Q}^2(\tilde{\mathbf{P}}), \tilde{Q}^3(\tilde{\mathbf{P}})]. \tag{3.2}$$

It is more convenient, however, to relate points of the deformed configuration to points of the reference *placement* $\mathcal{S}$. Each point $\tilde{\mathbf{P}}$ of $\mathcal{C}$ is the image of some point $\mathbf{P}$ of $\mathcal{S}$ via the mapping $\tilde{\mathbf{P}} = \phi(\mathbf{P})$. This mapping can be used to express the actual deformation of the shell in terms of points of the reference placement, viz.,

$$\mathbf{p} = \tilde{f}(\tilde{\mathbf{P}}) = \tilde{f}(\phi(\mathbf{P})) = (\tilde{f} \circ \phi)(\mathbf{P}) \equiv f(\mathbf{P}), \tag{3.3}$$

where the functional composition $f \equiv \tilde{f} \circ \phi$ defines a mapping from the reference placement to the current configuration. The three mappings we have introduced are illustrated in Figure 4. Thus, cylindrical coordinates of points on the current configuration are given in terms of the cylindrical coordinates $Q^A \equiv \{R, \Theta, Z\}$ on the reference placement by

$$q^a(\mathbf{p}) = f^a[Q^1(\mathbf{P}), Q^2(\mathbf{P}), Q^3(\mathbf{P})], \tag{3.4}$$

analogous to (3.2).

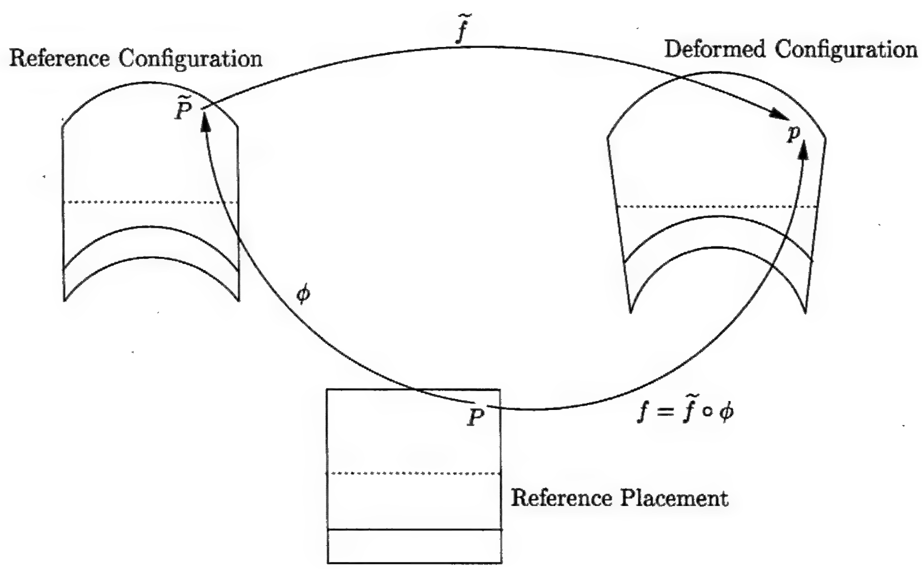

Figure 4: Mappings relating the reference placement, reference configuration, and deformed configuration.

The position vector of a material point in the reference configuration $\mathcal{S}$ of the coated membrane is given in terms of coordinates on the reference placement $\mathcal{C}$ by equation (2.8), repeated here:

$$\tilde{\mathbf{X}} = R\mathbf{E}_R + [\Gamma(R) + Z]\,\mathbf{E}_Z. \tag{3.5}$$

From (2.3), also repeated here:

$$\mathbf{E}_R = \cos\Theta\,\mathbf{i} + \sin\Theta\,\mathbf{j}, \quad \mathbf{E}_\Theta = -\sin\Theta\,\mathbf{i} + \cos\Theta\,\mathbf{j}, \quad \mathbf{E}_Z = \mathbf{k}, \tag{3.6}$$

we obtain the differentials of the basis vectors, expressed in terms of the same basis vectors:

$$d\mathbf{E}_R = d\Theta\mathbf{E}_\Theta, \quad d\mathbf{E}_\Theta = -d\Theta\mathbf{E}_R, \quad d\mathbf{E}_Z = 0. \tag{3.7}$$

Using these, we find for the differential of the position vector:

$$d\tilde{\mathbf{X}} = dR\,\mathbf{E}_R + Rd\Theta\mathbf{E}_\Theta + [dZ + \Gamma_{,R}dR]\,\mathbf{E}_Z. \tag{3.8}$$

In (3.8) we introduce the differential forms $\Omega_R = dR$, $\Omega_\Theta = Rd\Theta$, and $\Omega_Z = dZ$, to write it as

$$d\tilde{\mathbf{X}} = \Omega_R\,\mathbf{E}_R + \Omega_\Theta\mathbf{E}_\Theta + [\Omega_Z + \Gamma_{,R}\Omega_R]\,\mathbf{E}_Z = \Omega_A\,\mathbf{E}_A + \Gamma_{,R}\Omega_R\,\mathbf{E}_Z, \tag{3.9}$$

where the usual summation convention on repeated indices is to be understood, as in the first term of the second equality of (3.9), unless otherwise stated. Thus, the differential forms Ω_A can be written as $\Omega_A = H_A dQ^A$ (no sum on A), where $H_R = 1$, $H_\Theta = R$, and $H_Z = 1$ are referred to as scale factors.

The position vector of a point on the deformed configuration is given in terms of its cylindrical coordinates by

$$\mathbf{x} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + z\,\mathbf{k} = r\,\mathbf{e}_r + z\,\mathbf{e}_z, \tag{3.10}$$

where

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \quad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}, \quad \mathbf{e}_z = \mathbf{k}. \tag{3.11}$$

Similarly to (3.9), the differential of this position vector can be written as

$$d\mathbf{x} = \omega_a \mathbf{e}_a, \tag{3.12}$$

where $\omega_a = h_a dq^a$ (no sum on a) are differential forms on the deformed configuration (with $h_r = h_z = 1$, $h_\theta = r$).

From equation (3.4), we obtain

$$dq^a = \frac{\partial f^a}{\partial Q^A} dQ^A,$$

or, replacing coordinate differentials by their respective differential forms,

$$\omega_a = \frac{h_a}{H_A}\frac{\partial f^a}{\partial Q^A}\Omega_A \equiv F_{aA}\Omega_A, \tag{3.13}$$

where

$$F_{aA} \equiv \frac{h_a}{H_A}\frac{\partial f^a}{\partial Q^A} \qquad \text{(no sum on either } a \text{ or } A\text{)}, \tag{3.14}$$

are the elements of the matrix F of *deformation gradients* from the differential forms on the reference placement to differential forms on the deformed configuration.

The displacement of point $\widetilde{\mathbf{P}}$ of the reference configuration to point $\mathbf{p}$ of the deformed configuration is defined by the vector field

$$\mathbf{u} \equiv \mathbf{x} - \widetilde{\mathbf{X}}. \tag{3.15}$$

From this relation we obtain

$$d\mathbf{x} = d\widetilde{\mathbf{X}} + d\mathbf{u}. \tag{3.16}$$

If we write the displacement field $\mathbf{u}$ in terms of its components U_A in the orthonormal cylindrical basis of the reference placement, i.e.,

$$\mathbf{u} = U_R\,\mathbf{E}_R + U_\Theta\,\mathbf{E}_\Theta + U_Z\,\mathbf{E}_Z, \tag{3.17}$$

where the components U_A are assumed to be functions of the reference placement coordinates $\{R, \Theta, Z\}$, we find for its differential:

$$\begin{aligned} d\mathbf{u} &= (U_{R,R}dR + U_{R,\Theta}d\Theta + U_{R,Z}dZ)\,\mathbf{E}_R + U_R\,d\mathbf{E}_R \\ &\quad + (U_{\Theta,R}dR + U_{\Theta,\Theta}d\Theta + U_{\Theta,Z}dZ)\,\mathbf{E}_\Theta + U_\Theta\,d\mathbf{E}_\Theta \\ &\quad + (U_{Z,R}dR + U_{Z,\Theta}d\Theta + U_{Z,Z}dZ)\,\mathbf{E}_Z, \\ &= \left[U_{R,R}\Omega_R + \left(\frac{U_{R,\Theta} - U_\Theta}{R}\right)\Omega_\Theta + U_{R,Z}\Omega_Z\right]\mathbf{E}_R \\ &\quad + \left[U_{\Theta,R}\Omega_R + \left(\frac{U_{\Theta,\Theta} + U_R}{R}\right)\Omega_\Theta + U_{\Theta,Z}\Omega_Z\right]\mathbf{E}_\Theta \\ &\quad + \left[U_{Z,R}\Omega_R + \left(\frac{U_{Z,\Theta}}{R}\right)\Omega_\Theta + U_{Z,Z}\Omega_Z\right]\mathbf{E}_Z, \end{aligned} \tag{3.18}$$

where $U_{A,B}$ denotes the partial derivative of U_A with respect to Q_B. This can be written, using the summation convention, as

$$d\mathbf{u} = H_{AB}\Omega_B\mathbf{E}_A, \tag{3.19}$$

where $H_{AB} \equiv U_{A;B}$ are the elements of the matrix H containing the components $U_{A;B}$ of the *covariant* derivative of $\mathbf{u}$, distinguished by a semicolon in place of the comma. Explicitly, we have

$$H = \begin{bmatrix} U_{R,R} & (U_{R,\Theta} - U_\Theta)/R & U_{R,Z} \\ U_{\Theta,R} & (U_{\Theta,\Theta} + U_R)/R & U_{\Theta,Z} \\ U_{Z,R} & U_{Z,\Theta}/R & U_{Z,Z} \end{bmatrix}. \tag{3.20}$$

The Green-Lagrange strain tensor $\mathbf{E}$ is defined by

$$\mathbf{E} \equiv \frac{1}{2}\left(d\mathbf{x}\cdot d\mathbf{x} - d\widetilde{\mathbf{X}}\cdot d\widetilde{\mathbf{X}}\right) = \frac{1}{2}\left(d\widetilde{\mathbf{X}}\cdot d\mathbf{u} + d\mathbf{u}\cdot d\widetilde{\mathbf{X}} + d\mathbf{u}\cdot d\mathbf{u}\right), \tag{3.21}$$

where (3.16) was used to get the second equality. From (3.9) and (3.19), we have

$$d\widetilde{\mathbf{X}}\cdot d\mathbf{u} = (\Omega_C\,\mathbf{E}_C + \Gamma_{,R}\Omega_R\,\mathbf{E}_Z)\cdot H_{AB}\Omega_B\mathbf{E}_A = (H_{AB} + \Gamma_{,R}H_{ZB}\delta_{AR})\Omega_A\Omega_B,$$

after relabeling of dummy summation indices, and similarly,

$$d\mathbf{u}\cdot d\widetilde{\mathbf{X}} = H_{AB}\Omega_B\mathbf{E}_A\cdot(\Omega_C\,\mathbf{E}_C + \Gamma_{,R}\Omega_R\,\mathbf{E}_Z) = (H_{BA} + \Gamma_{,R}H_{ZA}\delta_{BR})\Omega_A\Omega_B.$$

Substitution of the last two results in (3.21) yields

$$\mathbf{E} = \frac{1}{2}(H_{AB} + H_{BA} + \Gamma_{,R}H_{ZB}\delta_{AR} + \Gamma_{,R}H_{ZA}\delta_{BR} + H_{CA}H_{CB})\,\Omega_A\Omega_B \equiv E_{AB}\,\Omega_A\Omega_B, \tag{3.22}$$

where $E_{AB} \equiv (H_{AB} + H_{BA} + \Gamma_{,R}H_{ZB}\delta_{AR} + \Gamma_{,R}H_{ZA}\delta_{BR} + H_{CA}H_{CB})/2$ are the elements of the Green-Lagrange strain matrix E. Carrying out the algebra, we obtain the following expressions for the components of the strain tensor in cylindrical coordinates on the reference placement:

$$E_{RR} = U_{R,R} + \Gamma_{,R}U_{Z,R} + \frac{1}{2}\left(U_{R,R}^2 + U_{\Theta,R}^2 + U_{Z,R}^2\right), \tag{3.23}$$

$$E_{\Theta\Theta} = \frac{U_{\Theta,\Theta} + U_R}{R} + \frac{1}{2}\left[\frac{(U_{R,\Theta} - U_\Theta)^2 + (U_{\Theta,\Theta} + U_R)^2 + U_{Z,\Theta}^2}{R^2}\right], \tag{3.24}$$

$$E_{ZZ} = U_{Z,Z} + \frac{1}{2}\left(U_{R,Z}^2 + U_{\Theta,Z}^2 + U_{Z,Z}^2\right), \tag{3.25}$$

$$E_{R\Theta} = \frac{1}{2}\left[U_{\Theta,R} + \Gamma_{,R}\frac{U_{Z,\Theta}}{R} + \frac{U_{R,\Theta} - U_\Theta}{R} + \frac{U_{R,R}(U_{R,\Theta} - U_\Theta) + U_{\Theta,R}(U_{\Theta,\Theta} + U_R) + U_{Z,R}U_{Z,\Theta}}{R}\right], \tag{3.26}$$

$$E_{\Theta Z} = \frac{1}{2}\left[U_{\Theta,Z} + \frac{U_{Z,\Theta}}{R} + \frac{U_{R,Z}(U_{R,\Theta} - U_\Theta) + U_{\Theta,Z}(U_{\Theta,\Theta} + U_R) + U_{Z,Z}U_{Z,\Theta}}{R}\right], \tag{3.27}$$

$$E_{RZ} = \frac{1}{2}(U_{R,Z} + U_{Z,R} + \Gamma_{,R}U_{Z,Z} + U_{R,R}U_{R,Z} + U_{\Theta,R}U_{\Theta,Z} + U_{Z,R}U_{Z,Z}), \tag{3.28}$$

where $E_{\Theta R} = E_{R\Theta}$, $E_{Z\Theta} = E_{\Theta Z}$, and $E_{ZR} = E_{RZ}$, i.e., E is *symmetric*. It is important to note that we are here taking over from the classical theory of laminates the fundamental assumption that the displacement components, hence the strain tensor components, are continuous through the coated membrane shell laminate.

Using (3.12), (3.13), (3.9), and (3.19) in (3.16), we obtain

$$F_{aB}\Omega_B\,\mathbf{e}_a = \Omega_A\,\mathbf{E}_A + \Gamma_{,R}\Omega_R\,\mathbf{E}_Z + H_{AB}\Omega_B\mathbf{E}_A = (\delta_{AB} + \Gamma_{,R}\delta_{AZ}\delta_{BR} + H_{AB})\,\Omega_B\mathbf{E}_A,$$

from which follows the useful relation:

$$\mathbf{e}_a = (\delta_{AB} + \Gamma_{,R}\delta_{AZ}\delta_{BR} + H_{AB})\,(F^{-1})_{Ba}\mathbf{E}_A \equiv K_{AB}(F^{-1})_{Ba}\mathbf{E}_A \equiv \mathcal{O}_{Aa}\mathbf{E}_A, \tag{3.29}$$

where

$$K_{AB} \equiv \delta_{AB} + \Gamma_{,R}\delta_{AZ}\delta_{BR} + H_{AB}, \qquad \mathcal{O}_{Aa} \equiv \mathbf{E}_A\cdot\mathbf{e}_a = K_{AB}(F^{-1})_{Ba}. \tag{3.30}$$

The matrix $\mathcal{O}$ with elements defined by (3.30) must be orthogonal, satisfying $\mathcal{O}^T\mathcal{O} = \mathcal{O}\mathcal{O}^T = I$ (where the T-superscript denotes a transposed matrix), since both bases $\{\mathbf{e}_a\}$ and $\{\mathbf{E}_A\}$ are orthonormal. It is an example of a *shifter* [3, p. 9], in this case from one orthonormal basis to another. Assuming both bases to have been chosen as right-handed, the determinant of $\mathcal{O}$ must be 1, i.e., $\det(\mathcal{O}) = \det(\mathcal{O}^T) = 1$. From the matrix form of the second equation of (3.30), i.e., $\mathcal{O} = KF^{-1} \Rightarrow F = \mathcal{O}^T K$, it then follows that

$$J \equiv \det(F) = \det(K), \tag{3.31}$$

where $J \equiv \det(F)$ is the Jacobian determinant of the matrix of deformation gradients. It is easy to show from (3.6) and (3.11) that the shifter from the orthonormal cylindrical basis on the reference placement to the orthonormal cylindrical basis on the current configuration is given by

$$\mathcal{O} = \begin{bmatrix} \cos(\theta-\Theta) & \sin(\theta-\Theta) & 0 \\ -\sin(\theta-\Theta) & \cos(\theta-\Theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.32}$$

Since $F = \mathcal{O}^T K$ we obtain, using (3.30) and (3.32):

$$F = \begin{bmatrix} \cos(\theta-\Theta) & -\sin(\theta-\Theta) & 0 \\ \sin(\theta-\Theta) & \cos(\theta-\Theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+U_{R,R} & (U_{R,\Theta}-U_\Theta)/R & U_{R,Z} \\ U_{\Theta,R} & 1+(U_{\Theta,\Theta}+U_R)/R & U_{\Theta,Z} \\ \Gamma_{,R}+U_{Z,R} & U_{Z,\Theta}/R & 1+U_{Z,Z} \end{bmatrix}, \tag{3.33}$$

which yields

$$F_{rR} = (1+U_{R,R})\cos(\theta-\Theta) - U_{\Theta,R}\sin(\theta-\Theta), \tag{3.34}$$

$$F_{r\Theta} = \left(\frac{U_{R,\Theta}-U_\Theta}{R}\right)\cos(\theta-\Theta) - \left(1+\frac{U_{\Theta,\Theta}+U_R}{R}\right)\sin(\theta-\Theta), \tag{3.35}$$

$$F_{rZ} = U_{R,Z}\cos(\theta-\Theta) - U_{\Theta,Z}\sin(\theta-\Theta), \tag{3.36}$$

$$F_{\theta R} = (1+U_{R,R})\sin(\theta-\Theta) + U_{\Theta,R}\cos(\theta-\Theta), \tag{3.37}$$

$$F_{\theta\Theta} = \left(\frac{U_{R,\Theta}-U_\Theta}{R}\right)\sin(\theta-\Theta) + \left(1+\frac{U_{\Theta,\Theta}+U_R}{R}\right)\cos(\theta-\Theta), \tag{3.38}$$

$$F_{\theta Z} = U_{R,Z} \sin(\theta - \Theta) + U_{\Theta,Z} \cos(\theta - \Theta), \tag{3.39}$$

$$F_{zR} = \Gamma_{,R} + U_{Z,R}, \qquad F_{z\Theta} = \frac{U_{Z,\Theta}}{R}, \qquad F_{zZ} = 1 + U_{Z,Z}. \tag{3.40}$$

Note that $dv = JdV$, where dV is a volume element on the reference placement and dv is a volume element on the current configuration. In order for the matrix F of deformation gradients to be invertible, we disallow the possibility that $J = 0$. We also disallow the possibility that $J < 0$, as that would imply that a volume element could have a negative volume. Thus, we require that

$$J > 0, \tag{3.41}$$

where, from (3.30) and (3.31):

$$\begin{aligned} J =& 1 + U_{R,R} + U_{Z,Z} + \frac{U_{\Theta,\Theta} + U_R}{R} + \left(\frac{U_{\Theta,\Theta} + U_R}{R}\right) U_{Z,Z} - \frac{U_{Z,\Theta}}{R} U_{\Theta,Z} + U_{R,R} U_{Z,Z} \\ &+ \left(\frac{U_{\Theta,\Theta} + U_R}{R}\right) U_{R,R} + \left(\frac{U_{\Theta,\Theta} + U_R}{R}\right) U_{R,R} U_{Z,Z} - \frac{U_{Z,\Theta}}{R} U_{\Theta,Z} U_{R,R} \\ &- \left(\frac{U_{R,\Theta} - U_\Theta}{R}\right) U_{\Theta,R} - \left(\frac{U_{R,\Theta} - U_\Theta}{R}\right) U_{\Theta,R} U_{Z,Z} + \left(\frac{U_{R,\Theta} - U_\Theta}{R}\right) U_{\Theta,Z} \Gamma_{,R} \\ &+ \left(\frac{U_{R,\Theta} - U_\Theta}{R}\right) U_{\Theta,Z} U_{Z,R} - U_{R,Z} \Gamma_{,R} - U_{Z,R} U_{R,Z} - \left(\frac{U_{\Theta,\Theta} + U_R}{R}\right) U_{R,Z} \Gamma_{,R} \\ &- \left(\frac{U_{\Theta,\Theta} + U_R}{R}\right) U_{Z,R} U_{R,Z} + \frac{U_{Z,\Theta}}{R} U_{R,Z} U_{\Theta,R}. \end{aligned} \tag{3.42}$$

4 Equilibrium Equations

Equilibrium of a deformed body requires that both the net force and net moment of force on any part of the body vanish. The vanishing of the net moment is well-known to imply the symmetry of the Cauchy stress tensor $\boldsymbol{\sigma}$ (our notation for stress tensors follows that of References [4, pp. 134-136], and [5, Chapter 4]). Here, we begin with the force equilibrium equations written in terms of the Cauchy stress tensor, and then reformulate them in terms of the first and second Piola-Kirchhoff stress tensors. Let $\mathcal{P}$ denote the volume of an arbitrary part of the coated membrane in its plate-like *reference placement*, and denote by $\partial\mathcal{P}$ the boundary surface of this part. Under the deformation f defined by (3.3), $\mathcal{P}$ is mapped to $f(\mathcal{P})$, bounded by the surface $\partial f(\mathcal{P})$. In the presence of a gravitational body force $\mathbf{f}_g$, force equilibrium of the arbitrary deformed volume $f(\mathcal{P})$ requires that

$$\oint_{\partial f(\mathcal{P})} \boldsymbol{\sigma} \cdot \mathbf{n}\, da + \int_{f(\mathcal{P})} \mathbf{f}_g\, dv \equiv \oint_{\partial f(\mathcal{P})} \sigma_{ab}\, n_b\, \mathbf{e}_a\, da + \int_{f(\mathcal{P})} \rho\, \mathbf{g} dv = 0, \tag{4.1}$$

where $\boldsymbol{\sigma} = \sigma_{ab}\mathbf{e}_a\mathbf{e}_b$, $\mathbf{n} = n_c\mathbf{e}_c$ is the unit normal to the *deformed* surface, ρ is the mass density of the material, and $\mathbf{g} = g\,\mathbf{e}_z = g\,\mathbf{E}_Z$ is the gravitational acceleration (assumed to act in the "up" direction along the positive Z-axis in Figure 2), expressed in the orthonormal cylindrical bases. The surface integral can be written in terms of quantities on the reference placement using a version of Nanson's formula (see, for example, [6], p. 249, or [7], p. 88), viz.,

$$n_a da = J(F^{-1})_{Aa} N_A dA, \tag{4.2}$$

where the N_A are components in the orthonormal cylindrical basis of the unit normal $\mathbf{N}$ to the surface element of area dA in the *reference placement*. The volume integral is transformed to one over the reference

placement volume by the substitution $dv = JdV$, where dV is the reference placement volume element. Making these substitutions in (4.1) yields

$$\oint_{\partial f(\mathcal{P})} \sigma_{ab}\, n_b\, \mathbf{e}_a\, da + \int_{f(\mathcal{P})} \rho\, g\, \mathbf{e}_z dv = \oint_{\partial \mathcal{P}} \sigma_{ab}\, J(F^{-1})_{Ab} N_A \mathbf{e}_a dA + \int_{f(\mathcal{P})} J\rho\, g\, \mathbf{e}_z dV$$
$$\equiv \oint_{\partial \mathcal{P}} P_{aA} N_A \mathbf{e}_a dA + \int_{f(\mathcal{P})} \rho_0\, g\, \mathbf{e}_z dV = 0, \quad (4.3)$$

where $\rho_0 \equiv J\rho$ is the mass density of material in the reference configuration, and

$$P_{aA} \equiv J\sigma_{ab}(F^{-1})_{Ab}, \quad (4.4)$$

are the components of the nonsymmetric *first Piola-Kirchhoff* stress tensor $\mathbf{P}$, which appears naturally in transforming from deformed configuration surface elements to reference placement surface elements. However, (4.3) is expressed in term of components along the *deformed* configuration basis vectors $\mathbf{e}_a$. The second Piola-Kirchhoff stress tensor $\mathbf{S}$ arises naturally by using (3.29) to *shift* to components along the orthonormal cylindrical basis vectors $\mathbf{E}_B$ of the *reference placement*, obtaining for the surface integral in (4.3):

$$\oint_{\partial \mathcal{P}} P_{aA} N_A \mathbf{e}_a dA = \oint_{\partial \mathcal{P}} P_{aA} N_A K_{BC} (F^{-1})_{Ca} \mathbf{E}_B dA \equiv \oint_{\partial \mathcal{P}} K_{BC} S_{CA} N_A \mathbf{E}_B dA, \quad (4.5)$$

where we identify

$$S_{CA} \equiv (F^{-1})_{Ca}\, P_{aA} = J\,(F^{-1})_{Ca}\, \sigma_{ab}\, (F^{-1})_{Ab} \quad (4.6)$$

as the components of the *second Piola-Kirchhoff* stress tensor. Thus, we can write (4.5) as

$$\oint_{\partial \mathcal{P}} P_{aA} N_A \mathbf{e}_a dA = \oint_{\partial \mathcal{P}} K_{BC} S_{CA} N_A \mathbf{E}_B dA \equiv \oint_{\partial \mathcal{P}} T_{BA} N_A \mathbf{E}_B dA, \quad (4.7)$$

where it was convenient to introduce yet another (nonsymmetric) stress tensor $\mathbf{T}$ with components defined by

$$T_{BA} \equiv K_{BC} S_{CA} = (\delta_{BC} + \Gamma_{,R} \delta_{BZ} \delta_{CR} + H_{BC}) S_{CA}. \quad (4.8)$$

Using (4.7) in the force equilibrium equations (4.3) yields

$$\oint_{\partial \mathcal{P}} T_{BA} N_A \mathbf{E}_B dA + \int_{f(\mathcal{P})} \rho_0\, g\, \mathbf{E}_Z dV = 0. \quad (4.9)$$

From (4.9) we infer that, just as the equations of equilibrium in terms of the Cauchy stress follow by an application of the divergence theorem to convert the surface integral on the deformed configuration to a volume integral, yielding from (4.1) local equilibrium equations of the form

$$\sigma_{ab;b} + \rho g\, \delta_{az} = 0, \quad (4.10)$$

where the left-hand side includes the covariant divergence of $\boldsymbol{\sigma}$ in cylindrical coordinates on the deformed configuration, so also must the nonsymmetric stress tensor $\boldsymbol{T}$ satisfy local equilibrium equations of the form

$$T_{AB;B} + \rho_0 g\, \delta_{AZ} = 0, \quad (4.11)$$

where the left-hand side includes the covariant divergence of $\boldsymbol{T}$ in cylindrical coordinates on the reference placement. Taking into account the *nonsymmetric* nature of the components T_{AB}, the component equations

of equilibrium in the radial, circumferential, and axial directions, respectively, thus have the same *form* as those for the Cauchy stress in cylindrical coordinates (see, for example, Reference [8], p. 306):

$$T_{RR,R} + \frac{1}{R}T_{R\Theta,\Theta} + T_{RZ,Z} + \frac{1}{R}(T_{RR} - T_{\Theta\Theta}) = 0 \quad \ldots\text{(radial)}, \tag{4.12}$$

$$T_{\Theta R,R} + \frac{1}{R}T_{\Theta\Theta,\Theta} + T_{\Theta Z,Z} + \frac{1}{R}(T_{\Theta R} + T_{R\Theta}) = 0 \quad \ldots\text{(circumferential)}, \tag{4.13}$$

$$T_{ZR,R} + \frac{1}{R}T_{Z\Theta,\Theta} + T_{ZZ,Z} + \frac{1}{R}T_{ZR} + \rho_0 g = 0 \quad \ldots\text{(axial)}. \tag{4.14}$$

The definition (4.8) has the matrix form $T = KS$, hence can be written as

$$T = \begin{bmatrix} 1 + U_{R,R} & (U_{R,\Theta} - U_\Theta)/R & U_{R,Z} \\ U_{\Theta,R} & 1 + (U_{\Theta,\Theta} + U_R)/R & U_{\Theta,Z} \\ \Gamma_{,R} + U_{Z,R} & U_{Z,\Theta}/R & 1 + U_{Z,Z} \end{bmatrix} \begin{bmatrix} S_{RR} & S_{R\Theta} & S_{RZ} \\ S_{R\Theta} & S_{\Theta\Theta} & S_{\Theta Z} \\ S_{RZ} & S_{\Theta Z} & S_{ZZ} \end{bmatrix}, \tag{4.15}$$

where we have used the symmetry of the second Piola-Kirchhoff stress tensor in writing this. Carrying out the matrix multiplication, we obtain the following expressions for the components of **T** in terms of the components of **S** and the displacement vector components and their derivatives:

$$T_{RR} = (1 + U_{R,R})\, S_{RR} + \frac{1}{R}(U_{R,\Theta} - U_\Theta)\, S_{R\Theta} + U_{R,Z}\, S_{RZ}, \tag{4.16}$$

$$T_{R\Theta} = (1 + U_{R,R})\, S_{R\Theta} + \frac{1}{R}(U_{R,\Theta} - U_\Theta)\, S_{\Theta\Theta} + U_{R,Z}\, S_{\Theta Z}, \tag{4.17}$$

$$T_{RZ} = (1 + U_{R,R})\, S_{RZ} + \frac{1}{R}(U_{R,\Theta} - U_\Theta)\, S_{\Theta Z} + U_{R,Z}\, S_{ZZ}, \tag{4.18}$$

$$T_{\Theta R} = U_{\Theta,R}\, S_{RR} + \left[1 + \frac{1}{R}(U_{\Theta,\Theta} + U_R)\right] S_{R\Theta} + U_{\Theta,Z}\, S_{RZ}, \tag{4.19}$$

$$T_{\Theta\Theta} = U_{\Theta,R}\, S_{R\Theta} + \left[1 + \frac{1}{R}(U_{\Theta,\Theta} + U_R)\right] S_{\Theta\Theta} + U_{\Theta,Z}\, S_{\Theta Z}, \tag{4.20}$$

$$T_{\Theta Z} = U_{\Theta,R}\, S_{RZ} + \left[1 + \frac{1}{R}(U_{\Theta,\Theta} + U_R)\right] S_{\Theta Z} + U_{\Theta,Z}\, S_{ZZ}, \tag{4.21}$$

$$T_{ZR} = (\Gamma_{,R} + U_{Z,R})\, S_{RR} + \frac{1}{R}U_{Z,\Theta}\, S_{R\Theta} + (1 + U_{Z,Z})\, S_{RZ}, \tag{4.22}$$

$$T_{Z\Theta} = (\Gamma_{,R} + U_{Z,R})\, S_{R\Theta} + \frac{1}{R}U_{Z,\Theta}\, S_{\Theta\Theta} + (1 + U_{Z,Z})\, S_{\Theta Z}, \tag{4.23}$$

$$T_{ZZ} = (\Gamma_{,R} + U_{Z,R})\, S_{RZ} + \frac{1}{R}U_{Z,\Theta}\, S_{\Theta Z} + (1 + U_{Z,Z})\, S_{ZZ}. \tag{4.24}$$

5 Constitutive Relations

Prior to applying a coating, a non-mechanical thermal stress due to CTE mismatch between mold and membrane develops as the two cool to room temperature following the annealing process. We denote this mismatch stress by S_s^{nm}. A coating of thickness h_c is then applied to the constrained membrane, and assumed to be perfectly bonded to it, hence fully constrained geometrically. The coating typically undergoes some microstructural change during the coating process, inducing an *intrinsic* coating stress S_c^{nm} (which may be either tensile or compressive). If the membrane is coated at a temperature different from the temperature at which the mold is stress-free, a thermal mismatch stress may also appear in the coating.

We assume that both materials are linearly elastic, uniform, homogeneous, and isotropic, and that they remain perfectly bonded after removal from the mold (guaranteeing continuity of the displacement components across their interface). As in [9], [10] and [11], we have begun with the full three-dimensional Green-Lagrange strain tensor E_{AB}, including all geometrically nonlinear terms involving displacement components or their partial derivatives, as shown in equations (3.23)–(3.28). The choice of the Green-Lagrange strain tensor dictates a *material* rather than *spatial* description of the deformation, and we assume that in each of the two materials we have a simple uniform, linear, isotropic constitutive relation between the Green-Lagrange strain tensor and the second Piola-Kirchhoff stress tensor:

$$S_{ABi} = S_i^{nm}\,\delta_{AB} + \frac{E_i}{(1+\nu_i)}\left[E_{AB} + \frac{\nu_i}{1-2\nu_i}\,(E_{RRi} + E_{\Theta\Theta i} + E_{ZZi})\,\delta_{AB}\right], \tag{5.1}$$

where the subscript i denotes the ith layer of the laminate, and the constants E_i and ν_i are Young's modulus and Poisson's ratio, respectively, of the component materials. The first term in (5.1) is a simple way of including the residual or non-mechanical stress in material i, which is assumed to be uniform and isotropic with constant value S_i^{nm}. It is a slight generalization of the constitutive relation given by Fung [2, pp. 354–355] intended to account for a *thermal* stress, in which case it is given by

$$S_i^{nm} = -\frac{E_i}{1-2\nu_i}\,\epsilon_i^{nm}, \tag{5.2}$$

where $\epsilon_i^{nm} = \alpha_i\,\Delta T$ is the thermal *strain*, α_i is the CTE of material i, and $\Delta T = T - T_0$ is the temperature deviation from some reference temperature T_0. It is assumed here to include possible *intrinsic* stresses in the materials so that, in general, the residual stress S_i^{nm} is a sum of intrinsic and thermal stresses. The component forms of the constitutive relations (5.1) can be written as

$$S_{RRi} = S_i^{nm} + \mathcal{E}_i\,[(1-\nu_i)\,E_{RR} + \nu_i\,(E_{\Theta\Theta} + E_{ZZ})], \tag{5.3}$$

$$S_{\Theta\Theta i} = S_i^{nm} + \mathcal{E}_i\,[(1-\nu_i)\,E_{\Theta\Theta} + \nu_i\,(E_{RR} + E_{ZZ})], \tag{5.4}$$

$$S_{ZZi} = S_i^{nm} + \mathcal{E}_i\,[(1-\nu_i)\,E_{ZZ} + \nu_i\,(E_{RR} + E_{\Theta\Theta})], \tag{5.5}$$

$$S_{R\Theta i} = G_i\,E_{R\Theta}, \qquad S_{RZi} = G_i\,E_{RZ}, \qquad S_{\Theta Zi} = G_i\,E_{\Theta Z}, \tag{5.6}$$

where we have introduced for convenience:

$$\mathcal{E}_i \equiv \frac{E_i}{(1+\nu_i)(1-2\nu_i)}, \qquad G_i \equiv \frac{E_i}{1+\nu_i}. \tag{5.7}$$

6 Boundary Condition of Pressure

To formulate appropriate boundary conditions for equations (4.12)-(4.14), we consider a shell that is in equilibrium under a difference in hydrostatic pressure on the *images* (under the deformation f) of the reference

placement faces $Z = h/2$ and $Z = -h/2$. Such a pressure is always normal to the *deformed* surfaces, and oppositely directed to their *outward* unit normal vectors. Thus, traction boundary conditions are most easily stated in terms of the Cauchy stress tensor on the deformed surfaces:

$$\boldsymbol{\sigma}^{\pm} \cdot \mathbf{n}^{\pm} = -p^{\pm}\,\mathbf{n}^{\pm} \quad \Rightarrow \quad \sigma^{\pm}_{ab} n^{\pm}_{b} da^{\pm} \mathbf{e}^{\pm}_{a} = -p^{\pm}\, n^{\pm}_{a} da^{\pm} \mathbf{e}^{\pm}_{a}, \tag{6.1}$$

where the $+$ and $-$ superscripts denote evaluations either on the planes $Z = h/2$ and $Z = -h/2$, respectively or, as in (6.1), on their *images* under f, and we have included for convenience the area elements $da^{\pm}$ in both sides of the second form of the boundary conditions. Using (3.29) and (4.2), the second equation of (6.1) can be reformulated in terms of quantities defined on the reference placement as follows:

$$\sigma^{\pm}_{ab} J^{\pm} (F^{-1})^{\pm}_{Ab}\, N^{\pm}_{A} dA^{\pm}\, K^{\pm}_{CB} (F^{-1})^{\pm}_{Ba} \mathbf{E}^{\pm}_{C} = -p^{\pm}\, J^{\pm} (F^{-1})^{\pm}_{Aa}\, N^{\pm}_{A} dA^{\pm}\, K^{\pm}_{CB} (F^{-1})^{\pm}_{Ba} \mathbf{E}^{\pm}_{C},$$

which reduces, after applying the definitions (4.6) and (4.8), to

$$T^{\pm}_{CA} N^{\pm}_{A} = -p^{\pm}\, J^{\pm} K^{\pm}_{CB} (F^{-1})^{\pm}_{Ba} (F^{-1})^{\pm}_{Aa} N^{\pm}_{A}. \tag{6.2}$$

The matrix products on the right-hand side of (6.2), viz., $KF^{-1}F^{-T}$, where F^{-T} is the transposed inverse of F, are easily manipulated to the identity $KF^{-1}F^{-T} \equiv K^{-T}$, using the relation $\mathcal{O} = KF^{-1}$ to replace F^{-1}, and the orthogonality of the shifter $\mathcal{O}$. Thus, after a relabeling of indices, the boundary conditions take the form

$$T^{\pm}_{AB} N^{\pm}_{B} = -p^{\pm}\, J^{\pm} (K^{-T})^{\pm}_{AB} N^{\pm}_{B}. \tag{6.3}$$

However, we note that the outward unit normals to the faces $Z = h/2$ and $Z = -h/2$ are $\mathbf{N}^{+} = \mathbf{E}_Z$ and $\mathbf{N}^{-} = -\mathbf{E}_Z$, respectively, so that $N^{\pm}_{B} = \pm\delta_{BZ}$ on *both sides* of (6.3), hence (6.3) reduces to

$$T^{\pm}_{AZ} = -p^{\pm}\, J^{\pm} (K^{-T})^{\pm}_{AZ}. \tag{6.4}$$

Recalling that $J = \det(F) = \det(K)$, it follows that $J^{\pm}(K^{-T})^{\pm}$ is just the matrix of cofactors of $K^{\pm}$. Thus, according to (6.4), elements of the *third columns* of the matrices $T^{\pm}$ and $-p^{\pm}J^{\pm}(K^{-T})^{\pm}$ must be identical, yielding the final forms of the boundary conditions of pressure:

$$T^{\pm}_{RZ} = -p^{\pm} \left[\frac{U^{\pm}_{Z,\Theta} U^{\pm}_{\Theta,R}}{R^{\pm}} - U^{\pm}_{Z,R} \left(1 + \frac{U^{\pm}_{R} + U^{\pm}_{\Theta,\Theta}}{R^{\pm}} \right) \right], \tag{6.5}$$

$$T^{\pm}_{\Theta Z} = -p^{\pm} \left[-\frac{U^{\pm}_{Z,\Theta}}{R^{\pm}} \left(1 + U^{\pm}_{R,R} \right) + U^{\pm}_{Z,R} \left(\frac{U^{\pm}_{R,\Theta} - U^{\pm}_{\Theta}}{R^{\pm}} \right) \right], \tag{6.6}$$

$$T^{\pm}_{ZZ} = -p^{\pm} \left[(1 + U^{\pm}_{R,R}) \left(1 + \frac{U^{\pm}_{R} + U^{\pm}_{\Theta,\Theta}}{R^{\pm}} \right) - U^{\pm}_{\Theta,R} \left(\frac{U^{\pm}_{R,\Theta} - U^{\pm}_{\Theta}}{R^{\pm}} \right) \right]. \tag{6.7}$$

7 On the Derivation of Theories of Two-Dimensional Elastic Bodies from the Three-Dimensional Theory of Elasticity, Using the Method of Asymptotic Expansions

We introduce dimensionless coordinates (ρ, ζ), and scaled displacement components $(\widehat{U}, \widehat{V}, \widehat{W})$, a scaled reference configuration function $\widehat{\Gamma}$, scaled stress components $\widehat{S}_{ABi}$, scaled residual in-plane stresses $\widehat{S}_{i}$, scaled pressure loads $\widehat{p}^{\pm}$, and a scaled gravitational body force $\rho_{0i}\widehat{g}$, defined by the following relations:

$$R = a\rho, \qquad Z = h\zeta = \varepsilon a \zeta, \tag{7.1}$$

$$\Gamma = \varepsilon^{r} a \widehat{\Gamma}, \qquad U_Z = \varepsilon^{m} a \widehat{W}, \qquad U_R = \varepsilon^{\ell} a \widehat{U}, \qquad U_\Theta = \varepsilon^{\ell} a \widehat{V}, \tag{7.2}$$

$$S_{RRi} = \varepsilon^n \Sigma_i \widehat{S}_{RRi}, \qquad S_{\Theta\Theta i} = \varepsilon^n \Sigma_i \widehat{S}_{\Theta\Theta i}, \qquad S_{R\Theta i} = \varepsilon^n \Sigma_i \widehat{S}_{R\Theta i}, \qquad S_i^{nm} = \varepsilon^n \Sigma_i \widehat{S}_i^{nm}, \tag{7.3}$$

$$S_{RZi} = \varepsilon^p \Sigma_i \widehat{S}_{RZi}, \qquad S_{\Theta Zi} = \varepsilon^p \Sigma_i \widehat{S}_{\Theta Zi}, \tag{7.4}$$

$$S_{ZZi} = \varepsilon^q \Sigma_i \widehat{S}_{ZZi}, \qquad p^{\pm} = \varepsilon^q \Sigma^{\pm} \widehat{p}^{\pm}, \qquad a\rho_{0i}\, g = \Sigma_i \varepsilon^t \widehat{g}, \tag{7.5}$$

where the Σ_i are two arbitrary constants with dimensions of stress ($\Sigma^+ = \Sigma_s$, and $\Sigma^- = \Sigma_c$), and $\varepsilon \equiv h/a$ is the (assumed to be small) scaling parameter. The exponent values are arbitrary at this point, but typically satisfy the inequalities $r \le m < \ell \le n < t \le p < q$. It should be noted that the original variables are functions of R, Θ, and Z, e.g., $U_R = U_R(R, \Theta, Z)$, while the scaled variables are all functions of ρ, Θ, and ζ, e.g., $\widehat{U} = \widehat{U}(\rho, \Theta, \zeta) \equiv (1/a\varepsilon^\ell) U_R(a\rho, \Theta, h_s\zeta)$. From these definitions, we obtain the following expressions for the partial derivatives of the displacement components, and the ordinary derivative of the surface-defining function Γ:

$$U_{R,R} = \varepsilon^\ell \widehat{U}_{,\rho}, \qquad U_{R,\Theta} = \varepsilon^\ell a \widehat{U}_{,\Theta}, \qquad U_{R,Z} = \varepsilon^{\ell-1} \widehat{U}_{,\zeta}, \tag{7.6}$$

$$U_{\Theta,R} = \varepsilon^\ell \widehat{V}_{,\rho}, \qquad U_{\Theta,\Theta} = \varepsilon^\ell a \widehat{V}_{,\Theta}, \qquad U_{\Theta,Z} = \varepsilon^{\ell-1} \widehat{V}_{,\zeta}, \tag{7.7}$$

$$U_{Z,R} = \varepsilon^m \widehat{W}_{,\rho}, \qquad U_{Z,\Theta} = \varepsilon^m a \widehat{W}_{,\Theta}, \qquad U_{Z,Z} = \varepsilon^{m-1} \widehat{W}_{,\zeta}, \qquad \Gamma_{,R} = \varepsilon^r \widehat{\Gamma}_{,\rho}. \tag{7.8}$$

Substitution of these expressions into equations (3.23)–(3.28) for the strain components yields:

$$E_{RR} = \varepsilon^\ell \widehat{U}_{,\rho} + \varepsilon^{r+m} \widehat{\Gamma}_{,\rho} \widehat{W}_{,\rho} + \frac{1}{2}\left[\varepsilon^{2m} \widehat{W}_{,\rho}^2 + \varepsilon^{2\ell}\left(\widehat{U}_{,\rho}^2 + \widehat{V}_{,\rho}^2\right)\right], \tag{7.9}$$

$$E_{\Theta\Theta} = \varepsilon^\ell \left(\frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho}\right) + \varepsilon^{2m} \frac{\widehat{W}_{,\Theta}^2}{2\rho^2} + \varepsilon^{2\ell}\left[\frac{(\widehat{U}_{,\Theta} - \widehat{V})^2 + (\widehat{V}_{,\Theta} + \widehat{U})^2}{2\rho^2}\right], \tag{7.10}$$

$$E_{ZZ} = \varepsilon^{m-1} \widehat{W}_{,\zeta} + \varepsilon^{2m-2} \frac{1}{2} \widehat{W}_{,\zeta}^2 + \varepsilon^{2\ell-2} \frac{1}{2}\left(\widehat{U}_{,\zeta}^2 + \widehat{V}_{,\zeta}^2\right), \tag{7.11}$$

$$E_{R\Theta} = \frac{1}{2}\Bigg\{\varepsilon^\ell \left(\widehat{V}_{,\rho} + \frac{\widehat{U}_{,\Theta} - \widehat{V}}{\rho}\right) + \varepsilon^{r+m} \frac{\widehat{\Gamma}_{,\rho} \widehat{W}_{,\Theta}}{\rho} + \varepsilon^{2m} \frac{\widehat{W}_{,\rho} \widehat{W}_{,\Theta}}{\rho} + \varepsilon^{2\ell}\left[\frac{\widehat{U}_{,\rho}(\widehat{U}_{,\Theta} - \widehat{V}) + \widehat{V}_{,\rho}(\widehat{V}_{,\Theta} + \widehat{U})}{\rho}\right]\Bigg\}, \tag{7.12}$$

$$E_{\Theta Z} = \frac{1}{2}\left\{\varepsilon^{\ell-1} \widehat{V}_{,\zeta} + \varepsilon^m \frac{\widehat{W}_{,\Theta}}{\rho} + \varepsilon^{2m-1} \frac{\widehat{W}_{,\zeta} \widehat{W}_{,\Theta}}{\rho} + \varepsilon^{2\ell-1}\left[\frac{\widehat{U}_{,\zeta}(\widehat{U}_{,\Theta} - \widehat{V}) + \widehat{V}_{,\zeta}(\widehat{V}_{,\Theta} + \widehat{U})}{\rho}\right]\right\}, \tag{7.13}$$

$$E_{RZ} = \frac{1}{2}\left[\varepsilon^{\ell-1} \widehat{U}_{,\zeta} + \varepsilon^m \widehat{W}_{,\rho} + \varepsilon^{r+m-1} \widehat{\Gamma}_{,\rho} \widehat{W}_{,\zeta} + \varepsilon^{2m-1} \widehat{W}_{,\rho} \widehat{W}_{,\zeta} + \varepsilon^{2\ell-1}\left(\widehat{U}_{,\rho} \widehat{U}_{,\zeta} + \widehat{V}_{,\rho} \widehat{V}_{,\zeta}\right)\right]. \tag{7.14}$$

Substitution of (7.1)–(7.5) in (3.34)–(3.40) yields for the scaled deformation gradient matrix elements:

$$F_{rR} = (1 + \varepsilon^\ell \widehat{U}_{,\rho}) \cos(\theta - \Theta) - \varepsilon^\ell \widehat{V}_{,\rho} \sin(\theta - \Theta), \tag{7.15}$$

$$F_{r\Theta} = \varepsilon^\ell \left(\frac{\widehat{U}_{,\Theta} - \widehat{V}}{\rho}\right) \cos(\theta - \Theta) - \left[1 + \varepsilon^\ell \left(\frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho}\right)\right] \sin(\theta - \Theta), \tag{7.16}$$

$$F_{rZ} = \varepsilon^{\ell-1} \widehat{U}_{,\zeta} \cos(\theta - \Theta) - \varepsilon^{\ell-1} \widehat{V}_{,\zeta} \sin(\theta - \Theta), \tag{7.17}$$

$$F_{\theta R} = (1 + \varepsilon^\ell \widehat{U}_{,\rho}) \sin(\theta - \Theta) + \varepsilon^\ell \widehat{V}_{,\rho} \cos(\theta - \Theta), \tag{7.18}$$

$$F_{\theta\Theta} = \varepsilon^{\ell}\left(\frac{\widehat{U}_{,\Theta}-\widehat{V}}{\rho}\right)\sin(\theta-\Theta)+\left[1+\varepsilon^{\ell}\left(\frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho}\right)\right]\cos(\theta-\Theta), \tag{7.19}$$

$$F_{\theta Z} = \varepsilon^{\ell-1}\widehat{U}_{,\varsigma}\sin(\theta-\Theta)+\varepsilon^{\ell-1}\widehat{V}_{,\varsigma}\cos(\theta-\Theta), \tag{7.20}$$

$$F_{zR} = \varepsilon^{r}\,\widehat{\Gamma}_{,\rho}+\varepsilon^{m}\,\widehat{W}_{,\rho}, \qquad F_{z\Theta} = \varepsilon^{m}\,\frac{\widehat{W}_{,\Theta}}{\rho}, \qquad F_{zZ} = 1+\varepsilon^{m-1}\,\widehat{W}_{,\varsigma}. \tag{7.21}$$

The scaled Jacobian determinant (3.42) takes the form

$$\begin{aligned}
J =& 1 + \varepsilon^{m-1}\,\widehat{W}_{,\varsigma} + \varepsilon^{\ell}\left(\widehat{U}_{,\rho}+\frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho}\right) + \varepsilon^{\ell+m-1}\left[\widehat{W}_{,\varsigma}\left(\widehat{U}_{,\rho}+\frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho}\right)-\widehat{W}_{,\rho}\widehat{U}_{,\varsigma}-\frac{\widehat{W}_{,\Theta}}{\rho}\widehat{V}_{,\varsigma}\right]\\
&+\varepsilon^{2\ell}\left[\widehat{U}_{,\rho}\left(\frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho}\right)-\widehat{V}_{,\rho}\left(\frac{\widehat{U}_{,\Theta}-\widehat{V}}{\rho}\right)\right] - \varepsilon^{\ell+r-1}\,\widehat{\Gamma}_{,\rho}\widehat{U}_{,\varsigma}\\
&+\varepsilon^{2\ell+m-1}\left[\left(\frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho}\right)\left(\widehat{U}_{,\rho}\widehat{W}_{,\varsigma}-\widehat{W}_{,\rho}\widehat{U}_{,\varsigma}\right)-\left(\frac{\widehat{U}_{,\Theta}-\widehat{V}}{\rho}\right)\left(\widehat{V}_{,\rho}\widehat{W}_{,\varsigma}-\widehat{W}_{,\rho}\widehat{V}_{,\varsigma}\right)\right.\\
&\left.+\frac{\widehat{W}_{,\Theta}}{\rho}\left(\widehat{V}_{,\rho}\widehat{U}_{,\varsigma}-\widehat{U}_{,\rho}\widehat{V}_{,\varsigma}\right)\right] + \varepsilon^{2\ell+r-1}\,\widehat{\Gamma}_{,\rho}\left[\left(\frac{\widehat{U}_{,\Theta}-\widehat{V}}{\rho}\right)\widehat{V}_{,\varsigma}-\left(\frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho}\right)\widehat{U}_{,\varsigma}\right].
\end{aligned} \tag{7.22}$$

The scaled expressions for the strain tensor components are to be substituted in the constitutive relations (5.3)–(5.6). We begin with (5.5) for S_{ZZ}, from which we obtain

$$\begin{aligned}
\varepsilon^{q}\Sigma_i\widehat{S}_{ZZi} =& \varepsilon^{n}\Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\Bigg\{(1-\nu_i)\left[\varepsilon^{m-1}\widehat{W}_{,\varsigma}+\varepsilon^{2m-2}\frac{1}{2}\widehat{W}^2_{,\varsigma}+\varepsilon^{2\ell-2}\,\frac{1}{2}\left(\widehat{U}^2_{,\varsigma}+\widehat{V}^2_{,\varsigma}\right)\right]\\
&+\nu_i\Bigg[\varepsilon^{\ell}\left(\widehat{U}_{,\rho}+\frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho}\right)+\varepsilon^{r+m}\,\widehat{\Gamma}_{,\rho}\widehat{W}_{,\rho}+\varepsilon^{2m}\left(\frac{1}{2}\widehat{W}^2_{,\rho}+\frac{\widehat{W}^2_{,\Theta}}{2\rho^2}\right)\\
&+\varepsilon^{2\ell}\left(\frac{\widehat{U}^2_{,\rho}+\widehat{V}^2_{,\rho}}{2}+\frac{(\widehat{U}_{,\Theta}-\widehat{V})^2+(\widehat{V}_{,\Theta}+\widehat{U})^2}{2\rho^2}\right)\Bigg]\Bigg\}.
\end{aligned} \tag{7.23}$$

The last two off-diagonal constitutive relations for the out-of-plane stress components yield the scaled relations:

$$\varepsilon^{p}\Sigma_i\widehat{S}_{RZi} = \frac{G_i}{2}\left[\varepsilon^{\ell-1}\,\widehat{U}_{,\varsigma}+\varepsilon^{m}\widehat{W}_{,\rho}+\varepsilon^{r+m-1}\widehat{\Gamma}_{,\rho}\widehat{W}_{,\varsigma}+\varepsilon^{2m-1}\widehat{W}_{,\rho}\widehat{W}_{,\varsigma}+\varepsilon^{2\ell-1}\left(\widehat{U}_{,\rho}\widehat{U}_{,\varsigma}+\widehat{V}_{,\rho}\widehat{V}_{,\varsigma}\right)\right], \tag{7.24}$$

$$\varepsilon^{p}\Sigma_i\widehat{S}_{\Theta Zi} = \frac{G_i}{2}\left\{\varepsilon^{\ell-1}\widehat{V}_{,\varsigma}+\varepsilon^{m}\frac{\widehat{W}_{,\Theta}}{\rho}+\varepsilon^{2m-1}\frac{\widehat{W}_{,\varsigma}\widehat{W}_{,\Theta}}{\rho}+\varepsilon^{2\ell-1}\left[\frac{\widehat{U}_{,\varsigma}(\widehat{U}_{,\Theta}-\widehat{V})+\widehat{V}_{,\varsigma}(\widehat{V}_{,\Theta}+\widehat{U})}{\rho}\right]\right\}, \tag{7.25}$$

respectively, and the scaled version of the off-diagonal in-plane constitutive relation (5.6) has the form

$$\begin{aligned}
\varepsilon^{n}\Sigma_i\widehat{S}_{R\Theta i} = \frac{G_i}{2}\Bigg\{&\varepsilon^{\ell}\left(\widehat{V}_{,\rho}+\frac{\widehat{U}_{,\Theta}-\widehat{V}}{\rho}\right)+\varepsilon^{r+m}\,\frac{\widehat{\Gamma}_{,\rho}\widehat{W}_{,\Theta}}{\rho}+\varepsilon^{2m}\frac{\widehat{W}_{,\rho}\widehat{W}_{,\Theta}}{\rho}\\
&+\varepsilon^{2\ell}\left[\frac{\widehat{U}_{,\rho}(\widehat{U}_{,\Theta}-\widehat{V})+\widehat{V}_{,\rho}(\widehat{V}_{,\Theta}+\widehat{U})}{\rho}\right]\Bigg\}.
\end{aligned} \tag{7.26}$$

The final two constitutive relations (5.3) and (5.4) take the following forms:

$$\begin{aligned}
\varepsilon^n \Sigma_i \widehat{S}_{RRi} = \varepsilon^n \Sigma_i \widehat{S}_i^{nm} + \mathcal{E}_i \Bigg((1-\nu_i) \left\{ \varepsilon^\ell \widehat{U}_{,\rho} + \varepsilon^{r+m} \widehat{\Gamma}_{,\rho} \widehat{W}_{,\rho} + \frac{1}{2} \left[\varepsilon^{2m} \widehat{W}_{,\rho}^2 + \varepsilon^{2\ell} \left(\widehat{U}_{,\rho}^2 + \widehat{V}_{,\rho}^2 \right) \right] \right\} \\
+ \nu_i \Bigg\{ \varepsilon^\ell \left(\frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho} \right) + \varepsilon^{2m} \frac{\widehat{W}_{,\Theta}^2}{2\rho^2} + \varepsilon^{2\ell} \left[\frac{(\widehat{U}_{,\Theta} - \widehat{V})^2 + (\widehat{V}_{,\Theta} + \widehat{U})^2}{2\rho^2} \right] \\
+ \varepsilon^{m-1} \widehat{W}_{,\varsigma} + \varepsilon^{2m-2} \frac{1}{2} \widehat{W}_{,\varsigma}^2 + \varepsilon^{2\ell-2} \frac{1}{2} \left(\widehat{U}_{,\varsigma}^2 + \widehat{V}_{,\varsigma}^2 \right) \Bigg\} \Bigg),
\end{aligned} \tag{7.27}$$

and

$$\begin{aligned}
\varepsilon^n \Sigma_i \widehat{S}_{\Theta\Theta i} = \varepsilon^n \Sigma_i \widehat{S}_i^{nm} + \mathcal{E}_i \Bigg((1-\nu_i) \left\{ \varepsilon^\ell \left(\frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho} \right) + \varepsilon^{2m} \frac{\widehat{W}_{,\Theta}^2}{2\rho^2} + \varepsilon^{2\ell} \left[\frac{(\widehat{U}_{,\Theta} - \widehat{V})^2 + (\widehat{V}_{,\Theta} + \widehat{U})^2}{2\rho^2} \right] \right\} \\
+ \nu_i \Bigg\{ \varepsilon^\ell \widehat{U}_{,\rho} + \varepsilon^{r+m} \widehat{\Gamma}_{,\rho} \widehat{W}_{,\rho} + \frac{1}{2} \left[\varepsilon^{2m} \widehat{W}_{,\rho}^2 + \varepsilon^{2\ell} \left(\widehat{U}_{,\rho}^2 + \widehat{V}_{,\rho}^2 \right) \right] \\
+ \varepsilon^{m-1} \widehat{W}_{,\varsigma} + \varepsilon^{2m-2} \frac{1}{2} \widehat{W}_{,\varsigma}^2 + \varepsilon^{2\ell-2} \frac{1}{2} \left(\widehat{U}_{,\varsigma}^2 + \widehat{V}_{,\varsigma}^2 \right) \Bigg\} \Bigg).
\end{aligned} \tag{7.28}$$

Next, applying the scalings obtained in (7.1)–(7.5), (7.6)–(7.8), and (7.9)–(7.14) to the right-hand sides of equations (4.16)–(4.24), we obtain the stress components T_{ABi} in terms of our scaled variables:

$$T_{RRi} = \varepsilon^n \Sigma_i \widehat{S}_{RRi} + \varepsilon^{n+\ell} \Sigma_i \left(\widehat{U}_{,\rho} \widehat{S}_{RRi} + \frac{\widehat{U}_{,\Theta} - \widehat{V}}{\rho} \widehat{S}_{R\Theta i} \right) + \varepsilon^{\ell+p-1} \Sigma_i \widehat{U}_{,\varsigma} \widehat{S}_{RZi}, \tag{7.29}$$

$$T_{R\Theta i} = \varepsilon^n \Sigma_i \widehat{S}_{R\Theta i} + \varepsilon^{n+\ell} \Sigma_i \left(\widehat{U}_{,\rho} \widehat{S}_{R\Theta i} + \frac{\widehat{U}_{,\Theta} - \widehat{V}}{\rho} \widehat{S}_{\Theta\Theta i} \right) + \varepsilon^{\ell+p-1} \Sigma_i \widehat{U}_{,\varsigma} \widehat{S}_{\Theta Zi}, \tag{7.30}$$

$$T_{RZi} = \varepsilon^p \Sigma_i \widehat{S}_{RZi} + \varepsilon^{p+\ell} \Sigma_i \left(\widehat{U}_{,\rho} \widehat{S}_{RZi} + \frac{\widehat{U}_{,\Theta} - \widehat{V}}{\rho} \widehat{S}_{\Theta Zi} \right) + \varepsilon^{\ell+q-1} \Sigma_i \widehat{U}_{,\varsigma} \widehat{S}_{ZZi}, \tag{7.31}$$

$$T_{\Theta Ri} = \varepsilon^n \Sigma_i \widehat{S}_{R\Theta i} + \varepsilon^{n+\ell} \Sigma_i \left(\widehat{V}_{,\rho} \widehat{S}_{RRi} + \frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho} \widehat{S}_{R\Theta i} \right) + \varepsilon^{\ell+p-1} \Sigma_i \widehat{V}_{,\varsigma} \widehat{S}_{RZi}, \tag{7.32}$$

$$T_{\Theta\Theta i} = \varepsilon^n \Sigma_i \widehat{S}_{\Theta\Theta i} + \varepsilon^{n+\ell} \Sigma_i \left(\widehat{V}_{,\rho} \widehat{S}_{R\Theta i} + \frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho} \widehat{S}_{\Theta\Theta i} \right) + \varepsilon^{\ell+p-1} \Sigma_i \widehat{V}_{,\varsigma} \widehat{S}_{\Theta Zi}, \tag{7.33}$$

$$T_{\Theta Zi} = \varepsilon^p \Sigma_i \widehat{S}_{\Theta Zi} + \varepsilon^{p+\ell} \Sigma_i \left(\widehat{V}_{,\rho} \widehat{S}_{RZi} + \frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho} \widehat{S}_{\Theta Zi} \right) + \varepsilon^{\ell+q-1} \Sigma_i \widehat{V}_{,\varsigma} \widehat{S}_{ZZi}, \tag{7.34}$$

$$T_{ZRi} = \varepsilon^p \Sigma_i \widehat{S}_{RZi} + \varepsilon^{r+n} \Sigma_i \widehat{\Gamma}_{,\rho} \widehat{S}_{RRi} + \varepsilon^{m+n} \Sigma_i \left(\widehat{W}_{,\rho} \widehat{S}_{RRi} + \frac{\widehat{W}_{,\Theta}}{\rho} \widehat{S}_{R\Theta i} \right) + \varepsilon^{m+p-1} \Sigma_i \widehat{W}_{,\varsigma} \widehat{S}_{RZi}, \tag{7.35}$$

$$T_{Z\Theta i} = \varepsilon^p \Sigma_i \widehat{S}_{\Theta Zi} + \varepsilon^{r+n} \Sigma_i \widehat{\Gamma}_{,\rho} \widehat{S}_{R\Theta i} + \varepsilon^{m+n} \Sigma_i \left(\widehat{W}_{,\rho} \widehat{S}_{R\Theta i} + \frac{\widehat{W}_{,\Theta}}{\rho} \widehat{S}_{\Theta\Theta i} \right) + \varepsilon^{m+p-1} \Sigma_i \widehat{W}_{,\varsigma} \widehat{S}_{\Theta Zi}, \tag{7.36}$$

$$T_{ZZi} = \varepsilon^q \Sigma_i \widehat{S}_{ZZi} + \varepsilon^{r+p} \Sigma_i \widehat{\Gamma}_{,\rho} \widehat{S}_{RZi} + \varepsilon^{m+p} \Sigma_i \left(\widehat{W}_{,\rho} \widehat{S}_{RZi} + \frac{\widehat{W}_{,\Theta}}{\rho} \widehat{S}_{\Theta Zi} \right) + \varepsilon^{m+q-1} \Sigma_i \widehat{W}_{,\varsigma} \widehat{S}_{ZZi}. \tag{7.37}$$

Substituting these scaled expressions in the equilibrium equations (4.12)–(4.14) yields equilibrium equations in terms of the scaled components of the second Piola-Kirchhoff stress tensor:

$$
\begin{aligned}
&\varepsilon^n\,\widehat{S}_{RRi,\rho} + \varepsilon^{n+\ell}\left(\widehat{U}_{,\rho}\widehat{S}_{RRi} + \frac{\widehat{U}_{,\Theta}-\widehat{V}}{\rho}\widehat{S}_{R\Theta i}\right)_{,\rho} + \varepsilon^{\ell+p-1}\left(\widehat{U}_{,\zeta}\widehat{S}_{RZi}\right)_{\rho}\\
&+\frac{1}{\rho}\left[\varepsilon^n\,\widehat{S}_{R\Theta i,\Theta} + \varepsilon^{n+\ell}\left(\widehat{U}_{,\rho}\widehat{S}_{R\Theta i} + \frac{\widehat{U}_{,\Theta}-\widehat{V}}{\rho}\widehat{S}_{\Theta\Theta i}\right)_{,\Theta} + \varepsilon^{\ell+p-1}\left(\widehat{U}_{,\zeta}\widehat{S}_{\Theta Zi}\right)_{,\Theta}\right]\\
&+\varepsilon^{p-1}\,\widehat{S}_{RZi,\zeta} + \varepsilon^{p+\ell-1}\left(\widehat{U}_{,\rho}\widehat{S}_{RZi} + \frac{\widehat{U}_{,\Theta}-\widehat{V}}{\rho}\widehat{S}_{\Theta Zi}\right)_{,\zeta} + \varepsilon^{\ell+q-2}\left(\widehat{U}_{,\zeta}\widehat{S}_{ZZi}\right)_{,\zeta}\\
&+\frac{1}{\rho}\Bigg\{\varepsilon^n\,\widehat{S}_{RRi} + \varepsilon^{n+\ell}\left(\widehat{U}_{,\rho}\widehat{S}_{RRi} + \frac{\widehat{U}_{,\Theta}-\widehat{V}}{\rho}\widehat{S}_{R\Theta i}\right) + \varepsilon^{\ell+p-1}\,\widehat{U}_{,\zeta}\widehat{S}_{RZi}\\
&-\left[\varepsilon^n\,\widehat{S}_{\Theta\Theta i} + \varepsilon^{n+\ell}\left(\widehat{V}_{,\rho}\widehat{S}_{R\Theta i} + \frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho}\widehat{S}_{\Theta\Theta i}\right) + \varepsilon^{\ell+p-1}\,\widehat{V}_{,\zeta}\widehat{S}_{\Theta Zi}\right]\Bigg\} = 0,
\end{aligned}
\tag{7.38}
$$

$$
\begin{aligned}
&\varepsilon^n\,\widehat{S}_{R\Theta i,\rho} + \varepsilon^{n+\ell}\left(\widehat{V}_{,\rho}\widehat{S}_{RRi} + \frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho}\widehat{S}_{R\Theta i}\right)_{,\rho} + \varepsilon^{\ell+p-1}\left(\widehat{V}_{,\zeta}\widehat{S}_{RZi}\right)_{\rho}\\
&+\frac{1}{\rho}\left[\varepsilon^n\,\widehat{S}_{\Theta\Theta i,\Theta} + \varepsilon^{n+\ell}\left(\widehat{V}_{,\rho}\widehat{S}_{R\Theta i} + \frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho}\widehat{S}_{\Theta\Theta i}\right)_{,\Theta} + \varepsilon^{\ell+p-1}\left(\widehat{V}_{,\zeta}\widehat{S}_{\Theta Zi}\right)_{,\Theta}\right]\\
&+\varepsilon^{p-1}\,\widehat{S}_{\Theta Zi,\zeta} + \varepsilon^{p+\ell-1}\left(\widehat{V}_{,\rho}\widehat{S}_{RZi} + \frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho}\widehat{S}_{\Theta Zi}\right)_{,\zeta} + \varepsilon^{\ell+q-2}\left(\widehat{V}_{,\zeta}\widehat{S}_{ZZi}\right)_{,\zeta}\\
&+\frac{1}{\rho}\Bigg\{\varepsilon^n\,\widehat{S}_{R\Theta i} + \varepsilon^{n+\ell}\left(\widehat{V}_{,\rho}\widehat{S}_{RRi} + \frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho}\widehat{S}_{R\Theta i}\right) + \varepsilon^{\ell+p-1}\,\widehat{V}_{,\zeta}\widehat{S}_{RZi}\\
&+\left[\varepsilon^n\,\widehat{S}_{R\Theta i} + \varepsilon^{n+\ell}\left(\widehat{U}_{,\rho}\widehat{S}_{R\Theta i} + \frac{\widehat{U}_{,\Theta}-\widehat{V}}{\rho}\widehat{S}_{\Theta\Theta i}\right) + \varepsilon^{\ell+p-1}\,\widehat{U}_{,\zeta}\widehat{S}_{\Theta Zi}\right]\Bigg\} = 0,
\end{aligned}
\tag{7.39}
$$

$$
\begin{aligned}
&\varepsilon^p\,\widehat{S}_{RZi,\rho} + \varepsilon^{r+n}\left(\widehat{\Gamma}_{,\rho}\widehat{S}_{RRi}\right)_{,\rho} + \varepsilon^{m+n}\left(\widehat{W}_{,\rho}\widehat{S}_{RRi} + \frac{\widehat{W}_{,\Theta}}{\rho}\widehat{S}_{R\Theta i}\right)_{,\rho} + \varepsilon^{m+p-1}\left(\widehat{W}_{,\zeta}\widehat{S}_{RZi}\right)_{,\rho}\\
&+\frac{1}{\rho}\left[\varepsilon^p\,\widehat{S}_{\Theta Zi,\Theta} + \varepsilon^{r+n}\left(\widehat{\Gamma}_{,\rho}\widehat{S}_{R\Theta i}\right)_{,\Theta} + \varepsilon^{m+n}\left(\widehat{W}_{,\rho}\widehat{S}_{R\Theta i} + \frac{\widehat{W}_{,\Theta}}{\rho}\widehat{S}_{\Theta\Theta i}\right)_{,\Theta} + \varepsilon^{m+p-1}\left(\widehat{W}_{,\zeta}\widehat{S}_{\Theta Zi}\right)_{,\Theta}\right]\\
&+\varepsilon^{q-1}\,\widehat{S}_{ZZi,\zeta} + \varepsilon^{r+p-1}\left(\widehat{\Gamma}_{,\rho}\widehat{S}_{RZi}\right)_{,\zeta} + \varepsilon^{m+p-1}\left(\widehat{W}_{,\rho}\widehat{S}_{RZi} + \frac{\widehat{W}_{,\Theta}}{\rho}\widehat{S}_{\Theta Zi}\right)_{,\zeta} + \varepsilon^{m+q-2}\left(\widehat{W}_{,\zeta}\widehat{S}_{ZZi}\right)_{,\zeta}\\
&+\frac{1}{\rho}\left[\varepsilon^p\,\widehat{S}_{RZi} + \varepsilon^{r+n}\,\widehat{\Gamma}_{,\rho}\widehat{S}_{RRi} + \varepsilon^{m+n}\left(\widehat{W}_{,\rho}\widehat{S}_{RRi} + \frac{\widehat{W}_{,\Theta}}{\rho}\widehat{S}_{R\Theta i}\right) + \varepsilon^{m+p-1}\,\widehat{W}_{,\zeta}\widehat{S}_{RZi}\right] + \varepsilon^t\,\widehat{g} = 0,
\end{aligned}
\tag{7.40}
$$

where the gravitational body force has been assumed, as stated earlier, to scale as follows:

$$a\rho_{0i}\,g = \Sigma_i\,\varepsilon^t\,\widehat{g}. \tag{7.41}$$

Finally, assuming that the pressures $p^{\pm}$ scale like the stress tensor component S_{ZZ}, i.e.,

$$p^{\pm} = \varepsilon^{q}\,\Sigma^{\pm}\,\widehat{p}^{\pm}, \tag{7.42}$$

our boundary conditions (6.5)-(6.7) take the following scaled forms, after eliminating common factors of $\Sigma^{\pm}$ on both sides of each equation:

$$\begin{aligned}
&\varepsilon^{p}\,\widehat{S}^{\pm}_{RZ} + \varepsilon^{p+\ell}\left[\widehat{U}^{\pm}_{,\rho}\widehat{S}^{\pm}_{RZ} + \left(\frac{\widehat{U}^{\pm}_{,\Theta} - \widehat{V}^{\pm}}{\rho}\right)\widehat{S}^{\pm}_{\Theta Z}\right] + \varepsilon^{\ell+q-1}\,\widehat{U}^{\pm}_{,\zeta}\widehat{S}^{\pm}_{ZZ} \\
&\qquad = \varepsilon^{q+m}\,\widehat{p}^{\pm}\,\widehat{W}^{\pm}_{,\rho} - \varepsilon^{q+m+\ell}\left[\frac{\widehat{W}^{\pm}_{,\Theta}\widehat{V}^{\pm}_{,\rho} - \widehat{W}^{\pm}_{,\rho}\left(\widehat{V}^{\pm}_{,\Theta} + \widehat{U}^{\pm}\right)}{\rho^{\pm}}\right]\widehat{p}^{\pm},
\end{aligned} \tag{7.43}$$

$$\begin{aligned}
&\varepsilon^{p}\,\widehat{S}^{\pm}_{\Theta Z} + \varepsilon^{p+\ell}\left[\widehat{V}^{\pm}_{,\rho}\widehat{S}^{\pm}_{RZ} + \left(\frac{\widehat{V}^{\pm}_{,\Theta} + \widehat{U}^{\pm}}{\rho^{\pm}}\right)\widehat{S}^{\pm}_{\Theta Z}\right] + \varepsilon^{q+\ell-1}\,\widehat{V}^{\pm}_{,\zeta}\widehat{S}^{\pm}_{ZZ} \\
&\qquad = \varepsilon^{q+m}\widehat{p}^{\pm}\frac{\widehat{W}^{\pm}_{,\Theta}}{\rho^{\pm}} + \varepsilon^{q+m+\ell}\left[\frac{\widehat{W}^{\pm}_{,\Theta}\widehat{U}^{\pm}_{,\rho} - \widehat{W}^{\pm}_{,\rho}\left(\widehat{U}^{\pm}_{,\Theta} - \widehat{V}^{\pm}\right)}{\rho^{\pm}}\right]\widehat{p}^{\pm},
\end{aligned} \tag{7.44}$$

$$\begin{aligned}
&\varepsilon^{q}\,\widehat{S}^{\pm}_{ZZ} + \varepsilon^{m+p}\left(\widehat{W}^{\pm}_{,\rho}\widehat{S}^{\pm}_{RZ} + \frac{\widehat{W}^{\pm}_{,\Theta}}{\rho^{\pm}}\widehat{S}^{\pm}_{\Theta Z}\right) + \varepsilon^{q+m-1}\,\widehat{W}^{\pm}_{,\zeta}\widehat{S}^{\pm}_{ZZ} \\
&= -\varepsilon^{q}\,\widehat{p}^{\pm} - \varepsilon^{q+\ell}\left(\widehat{U}^{\pm}_{,\rho} + \frac{\widehat{V}^{\pm}_{,\Theta} + \widehat{U}^{\pm}}{\rho^{\pm}}\right)\widehat{p}^{\pm} - \varepsilon^{q+2\ell}\left[\widehat{U}^{\pm}_{,\rho}\left(\frac{\widehat{V}^{\pm}_{,\Theta} + \widehat{U}^{\pm}}{\rho^{\pm}}\right) - \widehat{V}^{\pm}_{,\rho}\left(\frac{\widehat{U}^{\pm}_{,\Theta} - \widehat{V}^{\pm}}{\rho^{\pm}}\right)\right]\widehat{p}^{\pm}.
\end{aligned} \tag{7.45}$$

The method of asymptotic expansions proceeds from this point by introducing asymptotic series expansions in a new parameter $\delta = \varepsilon^{\mu}$ (where μ is yet another exponent) for each of the scaled stress tensor components, as well as each of the scaled displacement components and its partial derivatives. Thus, denoting by $\widehat{x}$ and $\widehat{y}$ any two such components (or partial derivatives of displacement components), we set

$$\widehat{x} = \sum_{k=0} \delta^{k}\,\widehat{x}_{(k)}, \qquad \widehat{y} = \sum_{k=0} \delta^{k}\,\widehat{y}_{(k)}, \qquad (\delta = \varepsilon^{\mu}). \tag{7.46}$$

Various theories of two-dimensional-like bodies are obtained by making particular, essentially *ad hoc*, choices of the scaling exponents in equations (7.2)–(7.5) and equation (7.46). For example, if we follow Tarn [12] and set $r = m = 1$, $\ell = n = 2$, $t = 3$, $p = 3$, $q = 4$, and $\mu = 2$, we obtain a generalized geometrically nonlinear, laminate shell theory (equivalent to von Kármán plate theory when specialized to a single material that is initially flat, so that $h_c = 0$ and $\Gamma(R) = 0$, and gravitational body forces are ignored). If, on the other hand, we follow the work of Erbay [11], and set $r = m = 1/2$, $\ell = n = 1$, $t = 3/2$, $p = 2$, $q = 5/2$, and $\mu = 1/2$, the leading order equations are a generalization of geometrically nonlinear membrane theory (equivalent to Hencky-Campbell membrane theory [13, 14] when specialized to a single material that is initially flat, and gravitational body forces are ignored). We have also found that by setting $r = 1$, $m = 2$, $\ell = n = 3$, $t = 4$, $p = 4$, $q = 5$, and $\mu = 1$, the leading order results are those of a geometrically linear, laminate shell theory (equivalent to classical laminate theory [15] when $\Gamma(R) = 0$), while $r = m = 1$, $\ell = n = 3/2$, $t = 5/2$, $p = 3$, $q = 7/2$, and $\mu = 1/2$, yields a theory of a geometrically linear membrane laminate. These characterizations of various theories by particular choices of the scaling exponents are tabulated in Table 1. In the Sections that follow, we develop each theory based on the choice of exponents indicated in this Table, where the terms "shell" and "membrane" distinguish the dependence of the leading order in-plane displacement components on the axial coordinate Z: for a membrane they are *independent* of Z, for a shell they are *linear* in Z.

Geometrically Nonlinear Shell	r	m	ℓ	n	t	p	q	μ
	1	1	2	2	3	3	4	2
Geometrically Nonlinear Membrane	r	m	ℓ	n	t	p	q	μ
	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{3}{2}$	2	$\frac{5}{2}$	$\frac{1}{2}$
Geometrically Linear Shell	r	m	ℓ	n	t	p	q	μ
	1	2	3	3	4	4	5	1
Geometrically Linear Membrane	r	m	ℓ	n	t	p	q	μ
	1	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	3	$\frac{7}{2}$	$\frac{1}{2}$

Table 1: Values of scaling exponents for various theories.

8 Geometrically Nonlinear Shell Laminate Theory

Following the work of Tarn [12] we set $r = m = 1$, $\ell = n = 2$, $t = p = 3$, and $q = 4$ in equations (7.2)–(7.5). Thus, we have

$$\Gamma = \varepsilon a\widehat{\Gamma}, \qquad U_Z = \varepsilon a\widehat{W}, \qquad U_R = \varepsilon^2 a\widehat{U}, \qquad U_\Theta = \varepsilon^2 a\widehat{V}, \tag{8.1}$$

$$S_{RRi} = \varepsilon^2\Sigma_i\widehat{S}_{RRi}, \qquad S_{\Theta\Theta i} = \varepsilon^2\Sigma_i\widehat{S}_{\Theta\Theta i}, \qquad S_{R\Theta i} = \varepsilon^2\Sigma_i\widehat{S}_{R\Theta i}, \qquad S_i^{nm} = \varepsilon^2\Sigma_i\widehat{S}_i^{nm}, \tag{8.2}$$

$$S_{RZi} = \varepsilon^3\Sigma_i\widehat{S}_{RZi}, \qquad S_{\Theta Zi} = \varepsilon^3\Sigma_i\widehat{S}_{\Theta Zi}, \tag{8.3}$$

$$S_{ZZi} = \varepsilon^4\Sigma_i\widehat{S}_{ZZi}, \qquad p^\pm = \varepsilon^4\,\Sigma^\pm\,\widehat{p}^\pm, \qquad a\rho_{0i}\,g = \Sigma_i\,\varepsilon^3\,\widehat{g}. \tag{8.4}$$

8.1 Leading Order Results Obtained by Scaling of the Constitutive Relations

Beginning with the constitutive relation (7.23) for S_{ZZ}, we obtain

$$\begin{aligned} \varepsilon^4\Sigma_i\widehat{S}_{ZZi} = \varepsilon^2\Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\Bigg\{ (1-\nu_i)\left[\widehat{W}_{,\varsigma} + \frac{1}{2}\widehat{W}_{,\varsigma}^2 + \varepsilon^2\left(\frac{\widehat{U}_{,\varsigma}^2 + \widehat{V}_{,\varsigma}^2}{2}\right)\right] \\ + \nu_i\left[\varepsilon^2\left(\widehat{U}_{,\rho} + \widehat{\Gamma}_{,\rho}\widehat{W}_{,\rho} + \frac{1}{2}\widehat{W}_{,\rho}^2 + \frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho} + \frac{\widehat{W}_{,\Theta}^2}{2\rho^2}\right)\right]\Bigg\} + O(\varepsilon^4). \end{aligned} \tag{8.5}$$

Again following Tarn [12], we set $\mu = 2$ in equation (7.46), and find for the *product* of any two of the asymptotic expansions (7.46), to tenth order in ε (recalling that $\delta = \varepsilon^\mu = \varepsilon^2$):

$$\begin{aligned} \widehat{xy} &= \sum_{k_1=0}\sum_{k_2=0} \delta^{(k_1+k_2)}\,\widehat{x}_{(k_1)}\,\widehat{y}_{(k_2)}, \\ &= \widehat{x}_{(0)}\widehat{y}_{(0)} + \left[\widehat{x}_{(0)}\widehat{y}_{(1)} + \widehat{x}_{(1)}\widehat{y}_{(0)}\right]\varepsilon^2 + \left[\widehat{x}_{(0)}\widehat{y}_{(2)} + \widehat{x}_{(1)}\widehat{y}_{(1)} + \widehat{x}_{(2)}\widehat{y}_{(0)}\right]\varepsilon^4 \\ &+ \left[\widehat{x}_{(0)}\widehat{y}_{(3)} + \widehat{x}_{(1)}\widehat{y}_{(2)} + \widehat{x}_{(2)}\widehat{y}_{(1)} + \widehat{x}_{(3)}\widehat{y}_{(0)}\right]\varepsilon^6 \\ &+ \left[\widehat{x}_{(0)}\widehat{y}_{(4)} + \widehat{x}_{(1)}\widehat{y}_{(3)} + \widehat{x}_{(2)}\widehat{y}_{(2)} + \widehat{x}_{(3)}\widehat{y}_{(1)} + \widehat{x}_{(4)}\widehat{y}_{(0)}\right]\varepsilon^8 \\ &+ \left[\widehat{x}_{(0)}\widehat{y}_{(5)} + \widehat{x}_{(1)}\widehat{y}_{(4)} + \widehat{x}_{(2)}\widehat{y}_{(3)} + \widehat{x}_{(3)}\widehat{y}_{(2)} + \widehat{x}_{(4)}\widehat{y}_{(1)} + \widehat{x}_{(5)}\widehat{y}_{(0)}\right]\varepsilon^{10} + O(\varepsilon^{12}). \end{aligned} \tag{8.6}$$

Substituting in equation (8.5) the asymptotic series for each of the variables, we obtain

$$\begin{aligned}\varepsilon^4\Sigma_i\widehat{S}_{(0)ZZi} = \varepsilon^2\Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\Bigg\{(1-\nu_i)\Bigg[\widehat{W}_{(0),\zeta} + \frac{1}{2}\widehat{W}^2_{(0),\zeta} + \varepsilon^2\Bigg(\widehat{W}_{(1),\zeta} + 2\widehat{W}_{(0),\zeta}\widehat{W}_{(1),\zeta} \\ + \frac{\widehat{U}^2_{(0),\zeta} + \widehat{V}^2_{(0),\zeta}}{2}\Bigg)\Bigg] + \nu_i\Bigg[\varepsilon^2\Bigg(\widehat{U}_{(0),\rho} + \widehat{\Gamma}_{,\rho}\widehat{W}_{(0),\rho} + \frac{1}{2}\widehat{W}^2_{(0),\rho} \\ + \frac{\widehat{V}_{(0),\Theta} + \widehat{U}_{(0)}}{\rho} + \frac{\widehat{W}^2_{(0),\Theta}}{2\rho^2}\Bigg)\Bigg]\Bigg\} + O(\varepsilon^4).\end{aligned} \tag{8.7}$$

The leading order term of this relation yields our first important result:

$$\widehat{W}_{(0),\zeta} + \frac{1}{2}\widehat{W}^2_{(0),\zeta} \equiv \frac{1}{2}\widehat{W}_{(0),\zeta}\left(2 + \widehat{W}_{(0),\zeta}\right) = 0. \tag{8.8}$$

Thus, $\widehat{W}_{(0),\zeta}$ must satisfy either $\widehat{W}_{(0),\zeta} = 0$, or $\widehat{W}_{(0),\zeta} = -2$. In order to eliminate the second possibility, we appeal to the form of the Jacobian determinant under the scalings being considered, viz., from (7.22):

$$\begin{aligned}J = 1 + \widehat{W}_{,\zeta} + \varepsilon^2\left[\widehat{U}_{,\rho} + \frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho} + \widehat{W}_{,\zeta}\left(\widehat{U}_{,\rho} + \frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho}\right) - \widehat{W}_{,\rho}\widehat{U}_{,\zeta} - \frac{\widehat{W}_{,\Theta}}{\rho}\widehat{V}_{,\zeta} - \widehat{\Gamma}_{,\rho}\widehat{U}_{,\zeta}\right] \\ + \varepsilon^4\left[\widehat{U}_{,\rho}\left(\frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho}\right) - \widehat{V}_{,\rho}\left(\frac{\widehat{U}_{,\Theta} - \widehat{V}}{\rho}\right) + \left(\frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho}\right)\left(\widehat{U}_{,\rho}\widehat{W}_{,\zeta} - \widehat{W}_{,\rho}\widehat{U}_{,\zeta} - \widehat{\Gamma}_{,\rho}\widehat{U}_{,\zeta}\right)\right. \\ \left. - \left(\frac{\widehat{U}_{,\Theta} - \widehat{V}}{\rho}\right)\left(\widehat{V}_{,\rho}\widehat{W}_{,\zeta} - \widehat{W}_{,\rho}\widehat{V}_{,\zeta} - \widehat{\Gamma}_{,\rho}\widehat{V}_{,\zeta}\right) + \frac{\widehat{W}_{,\Theta}}{\rho}\left(\widehat{V}_{,\rho}\widehat{U}_{,\zeta} - \widehat{U}_{,\rho}\widehat{V}_{,\zeta}\right)\right].\end{aligned} \tag{8.9}$$

In the limit $\varepsilon \to 0$, the condition $J > 0$ implies that we must have $\widehat{W}_{(0),\zeta} > -1$, which precludes the second possible solution $\widehat{W}_{(0),\zeta} = -2$. Thus, we must have

$$\widehat{W}_{(0),\zeta} = 0 \quad \Rightarrow \quad \widehat{W}_{(0)} = \widehat{w}(\rho, \Theta), \tag{8.10}$$

where $\widehat{w}$ is an arbitrary function of ρ and Θ only. Under these conditions, equation (8.7) reduces, after dividing through by ε^2, to

$$\begin{aligned}\varepsilon^2\Sigma_i\widehat{S}_{(0)ZZi} = \Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\Bigg[(1-\nu_i)\left(\widehat{W}_{(1),\zeta} + \frac{\widehat{U}^2_{(0),\zeta} + \widehat{V}^2_{(0),\zeta}}{2}\right) \\ + \nu_i\left(\widehat{U}_{(0),\rho} + \widehat{\Gamma}_{,\rho}\widehat{w}_{,\rho} + \frac{1}{2}\widehat{w}^2_{,\rho} + \frac{\widehat{V}_{(0),\Theta} + \widehat{U}_{(0)}}{\rho} + \frac{\widehat{w}^2_{,\Theta}}{2\rho^2}\right) + O(\varepsilon^2)\Bigg].\end{aligned} \tag{8.11}$$

The leading order term on the right-hand side of this equation yields another relation that will be needed later:

$$\widehat{W}_{(1),\zeta} + \frac{\widehat{U}^2_{(0),\zeta} + \widehat{V}^2_{(0),\zeta}}{2} = -\frac{\Sigma_i\widehat{S}_i^{nm}}{(1-\nu_i)\mathcal{E}_i} - \frac{\nu_i}{1-\nu_i}\left(\widehat{U}_{(0),\rho} + \widehat{\Gamma}_{,\rho}\widehat{w}_{,\rho} + \frac{1}{2}\widehat{w}^2_{,\rho} + \frac{\widehat{V}_{(0),\Theta} + \widehat{U}_{(0)}}{\rho} + \frac{\widehat{w}^2_{,\Theta}}{2\rho^2}\right). \tag{8.12}$$

Next, consider the two off-diagonal constitutive relations (7.24) and (7.25) for the out-of-plane stress components. With the present scalings, we obtain

$$\varepsilon^2\Sigma_i\widehat{S}_{RZi} = \frac{G_i}{2}\left[\left(\widehat{U}_{,\zeta} + \widehat{W}_{,\rho} + \widehat{\Gamma}_{,\rho}\widehat{W}_{,\zeta} + \widehat{W}_{,\rho}\widehat{W}_{,\zeta}\right) + \varepsilon^2\left(\widehat{U}_{,\rho}\widehat{U}_{,\zeta} + \widehat{V}_{,\rho}\widehat{V}_{,\zeta}\right)\right], \tag{8.13}$$

$$\varepsilon^2 \Sigma_i \widehat{S}_{\Theta Z i} = \frac{G_i}{2}\left[\left(\widehat{V}_{,\zeta} + \frac{\widehat{W}_{,\Theta}}{\rho} + \frac{\widehat{W}_{,\zeta}\widehat{W}_{,\Theta}}{\rho}\right) + \varepsilon^2\left(\frac{\widehat{U}_{,\zeta}(\widehat{U}_{,\Theta} - \widehat{V}) + \widehat{V}_{,\zeta}(\widehat{V}_{,\Theta} + \widehat{U})}{\rho}\right)\right], \tag{8.14}$$

where a common factor of ε was cancelled in both expressions. From the leading order terms of these expressions, recalling that $\widehat{W}_{(0),\zeta} = 0$, we obtain the following two equations:

$$\widehat{U}_{(0),\zeta} + \widehat{w}_{,\rho} = 0, \qquad \widehat{V}_{(0),\zeta} + \frac{\widehat{w}_{,\Theta}}{\rho} = 0,$$

which can be integrated to obtain

$$\widehat{U}_{(0)} = \widehat{u}(\rho,\Theta) - \zeta\,\widehat{w}_{,\rho}, \qquad \text{and} \qquad \widehat{V}_{(0)} = \widehat{v}(\rho,\Theta) - \zeta\frac{\widehat{w}_{,\Theta}}{\rho}, \tag{8.15}$$

where $\widehat{u}$ and $\widehat{v}$ are arbitrary functions of ρ and Θ only. Thus, the leading order results have provided non-dimensional forms of the well known Kirchhoff-Love expressions for the displacement components.

The scaled version of the off-diagonal in-plane constitutive relation (7.26) reduces under the present scalings to

$$\Sigma_i \widehat{S}_{R\Theta i} = \frac{G_i}{2}\left[\left(\widehat{V}_{,\rho} + \frac{\widehat{U}_{,\Theta} - \widehat{V}}{\rho} + \frac{\widehat{\Gamma}_{,\rho}\widehat{W}_{,\Theta}}{\rho} + \frac{\widehat{W}_{,\rho}\widehat{W}_{,\Theta}}{\rho}\right) + \varepsilon^2\left(\frac{\widehat{U}_{,\rho}(\widehat{U}_{,\Theta} - \widehat{V}) + \widehat{V}_{,\rho}(\widehat{V}_{,\Theta} + \widehat{U})}{\rho}\right)\right],$$

where a common factor of ε^2 was cancelled. Thus, to leading order we obtain

$$\Sigma_i \widehat{S}_{(0)R\Theta i} = \frac{G_i}{2}\left(\widehat{V}_{(0),\rho} + \frac{\widehat{U}_{(0),\Theta} - \widehat{V}_{(0)}}{\rho} + \frac{\widehat{\Gamma}_{,\rho}\widehat{w}_{,\Theta}}{\rho} + \frac{\widehat{w}_{,\rho}\widehat{w}_{,\Theta}}{\rho}\right).$$

Introducing an in-plane strain component $\widehat{\epsilon}_{R\Theta}$ defined by

$$\begin{aligned}
\widehat{\epsilon}_{R\Theta} &\equiv \frac{1}{2}\left(\widehat{V}_{(0),\rho} + \frac{\widehat{U}_{(0),\Theta} - \widehat{V}_{(0)}}{\rho} + \frac{\widehat{\Gamma}_{,\rho}\widehat{w}_{,\Theta}}{\rho} + \frac{\widehat{w}_{,\rho}\widehat{w}_{,\Theta}}{\rho}\right), \\
&= \frac{1}{2}\left[\widehat{v}_{,\rho} + \frac{\widehat{u}_{,\Theta} - \widehat{v}}{\rho} + \frac{\widehat{\Gamma}_{,\rho}\widehat{w}_{,\Theta}}{\rho} + \frac{\widehat{w}_{,\rho}\widehat{w}_{,\Theta}}{\rho} - 2\zeta\left(\frac{\widehat{w}_{,\rho\Theta}}{\rho} - \frac{\widehat{w}_{,\Theta}}{\rho^2}\right)\right], \\
&\equiv \widehat{\epsilon}^0_{R\Theta} - \zeta\,\widehat{k}_{R\Theta},
\end{aligned} \tag{8.16}$$

where we have introduced ζ-independent terms in the last line defined by

$$\widehat{\epsilon}^0_{R\Theta} \equiv \frac{1}{2}\left(\widehat{v}_{,\rho} + \frac{\widehat{u}_{,\Theta} - \widehat{v}}{\rho} + \frac{\widehat{\Gamma}_{,\rho}\widehat{w}_{,\Theta}}{\rho} + \frac{\widehat{w}_{,\rho}\widehat{w}_{,\Theta}}{\rho}\right), \qquad \widehat{k}_{R\Theta} \equiv \frac{\widehat{w}_{,\rho\Theta}}{\rho} - \frac{\widehat{w}_{,\Theta}}{\rho^2}, \tag{8.17}$$

we can write this constitutive relation as simply

$$\Sigma_i \widehat{S}_{(0)R\Theta i} = G_i\,\widehat{\epsilon}_{R\Theta}. \tag{8.18}$$

The final two constitutive relations (7.27) and (7.28) take the following forms under the present scalings:

$$\begin{aligned}
\varepsilon^2 \Sigma_i \widehat{S}_{RRi} = \varepsilon^2 \Sigma_i \widehat{S}^{nm}_i + \mathcal{E}_i\Bigg\{ &(1-\nu_i)\,\varepsilon^2\left(\widehat{U}_{,\rho} + \widehat{\Gamma}_{,\rho}\widehat{W}_{,\rho} + \frac{1}{2}\widehat{W}^2_{,\rho}\right) \\
+ \nu_i &\left[\varepsilon^2\left(\frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho} + \frac{\widehat{W}^2_{,\Theta}}{2\rho^2}\right) + \widehat{W}_{,\zeta} + \frac{1}{2}\widehat{W}^2_{,\zeta} + \varepsilon^2\left(\frac{\widehat{U}^2_{,\zeta} + \widehat{V}^2_{,\zeta}}{2}\right)\right]\Bigg\} + O(\varepsilon^4),
\end{aligned} \tag{8.19}$$

and

$$\varepsilon^2\Sigma_i\widehat{S}_{\Theta\Theta i} = \varepsilon^2\Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\Bigg\{ (1-\nu_i)\,\varepsilon^2 \left(\frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho} + \frac{\widehat{W}_{,\Theta}^2}{2\rho^2} \right) + \nu_i\left[\varepsilon^2\left(\widehat{U}_{,\rho} + \widehat{\Gamma}_{,\rho}\widehat{W}_{,\rho} + \frac{1}{2}\widehat{W}_{,\rho}^2 \right) + \widehat{W}_{,\zeta} + \frac{1}{2}\widehat{W}_{,\zeta}^2 + \varepsilon^2\left(\frac{\widehat{U}_{,\zeta}^2 + \widehat{V}_{,\zeta}^2}{2} \right) \right] \Bigg\} + O(\varepsilon^4). \tag{8.20}$$

The leading order terms involving $\widehat{W}_{,\zeta}$ in these expressions vanish according to (8.8), and (8.12) can be used to replace the second-order terms involving $\widehat{W}_{(1),\zeta}$, yielding

$$\Sigma_i\widehat{S}_{(0)RRi} = \Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\Bigg\{ (1-\nu_i)\,\widehat{\epsilon}_{RR} + \nu_i\left[\widehat{\epsilon}_{\Theta\Theta} - \frac{\Sigma_i\widehat{S}_i^{nm}}{(1-\nu_i)\mathcal{E}_i} - \frac{\nu_i}{1-\nu_i}\,(\widehat{\epsilon}_{RR}+\widehat{\epsilon}_{\Theta\Theta}) \right] \Bigg\}, \tag{8.21}$$

and

$$\Sigma_i\widehat{S}_{(0)\Theta\Theta i} = \Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\Bigg\{ (1-\nu_i)\,\widehat{\epsilon}_{\Theta\Theta} + \nu_i\left[\widehat{\epsilon}_{RR} - \frac{\Sigma_i\widehat{S}_i^{nm}}{(1-\nu_i)\mathcal{E}_i} - \frac{\nu_i}{1-\nu_i}\,(\widehat{\epsilon}_{RR}+\widehat{\epsilon}_{\Theta\Theta}) \right] \Bigg\}. \tag{8.22}$$

where we have introduced scaled in-plane strain components $\widehat{\epsilon}_{RR}$ and $\widehat{\epsilon}_{\Theta\Theta}$ defined by

$$\widehat{\epsilon}_{RR} \equiv \widehat{U}_{(0),\rho} + \widehat{\Gamma}_{,\rho}\widehat{w}_{,\rho} + \frac{1}{2}\widehat{w}_{,\rho}^2 = \widehat{u}_{,\rho} + \widehat{\Gamma}_{,\rho}\widehat{w}_{,\rho} + \frac{1}{2}\widehat{w}_{,\rho}^2 - \zeta\,\widehat{w}_{,\rho\rho} \equiv \widehat{\epsilon}_{RR}^0 - \zeta\,\widehat{k}_{RR}, \tag{8.23}$$

$$\widehat{\epsilon}_{\Theta\Theta} \equiv \frac{\widehat{V}_{(0),\Theta} + \widehat{U}_{(0)}}{\rho} + \frac{\widehat{w}_{,\Theta}^2}{2\rho^2} = \frac{\widehat{v}_{,\Theta}+\widehat{u}}{\rho} + \frac{\widehat{w}_{,\Theta}^2}{2\rho^2} - \zeta\left(\frac{\widehat{w}_{,\rho}}{\rho} + \frac{\widehat{w}_{,\Theta\Theta}}{\rho^2} \right) \equiv \widehat{\epsilon}_{\Theta\Theta}^0 - \zeta\,\widehat{k}_{\Theta\Theta}. \tag{8.24}$$

The ζ-independent terms appearing in the last two equations are defined by

$$\widehat{\epsilon}_{RR}^0 \equiv \widehat{u}_{,\rho} + \widehat{\Gamma}_{,\rho}\widehat{w}_{\rho} + \frac{1}{2}\widehat{w}_{,\rho}^2\,, \qquad \widehat{k}_{RR} \equiv \widehat{w}_{,\rho\rho}, \tag{8.25}$$

and

$$\widehat{\epsilon}_{\Theta\Theta}^0 \equiv \frac{\widehat{v}_{,\Theta}+\widehat{u}}{\rho} + \frac{\widehat{w}_{,\Theta}^2}{2\rho^2}\,, \qquad \widehat{k}_{\Theta\Theta} \equiv \frac{\widehat{w}_{,\rho}}{\rho} + \frac{\widehat{w}_{,\Theta\Theta}}{\rho^2}. \tag{8.26}$$

Replacing $\mathcal{E}_i$ by its definition (5.7) in (8.21) and (8.22), and simplifying the results, yields

$$\Sigma_i\widehat{S}_{(0)RRi} = \Sigma_i\widehat{S}_i + \frac{E_i}{1-\nu_i^2}\,(\widehat{\epsilon}_{RR} + \nu_i\widehat{\epsilon}_{\Theta\Theta}), \tag{8.27}$$

$$\Sigma_i\widehat{S}_{(0)\Theta\Theta i} = \Sigma_i\widehat{S}_i + \frac{E_i}{1-\nu_i^2}\,(\widehat{\epsilon}_{\Theta\Theta} + \nu_i\widehat{\epsilon}_{RR}), \tag{8.28}$$

where

$$\widehat{S}_i \equiv \frac{1-2\nu_i}{1-\nu_i}\widehat{S}_i^{nm}. \tag{8.29}$$

We conclude this Section by rewriting the important results in terms of leading order variables that are functions of the physical coordinates R, Θ, and Z, viz.,

$$u \equiv \varepsilon^2 a\widehat{u}, \qquad v \equiv \varepsilon^2 a\widehat{v}, \qquad w \equiv \varepsilon a\widehat{w} \tag{8.30}$$

$$U_{(0)Z} \equiv \varepsilon a\widehat{W}_{(0)} = w, \qquad U_{(0)R} \equiv \varepsilon^2 a\widehat{U}_{(0)} = u - Z\,w_{,R}, \qquad U_{(0)\Theta} \equiv \varepsilon^2 a\widehat{V}_{(0)} = v - Z\,\frac{w_{,\Theta}}{R}, \quad (8.31)$$

$$S_{(0)R\Theta i} \equiv \varepsilon^2 \Sigma_i \widehat{S}_{(0)R\Theta i} = G_i\,\epsilon_{R\Theta}, \quad (8.32)$$

$$S_{(0)RRi} \equiv \varepsilon^2 \Sigma_i \widehat{S}_{(0)RRi} = S_i + \frac{E_i}{1-\nu_i^2}\left(\epsilon_{RR} + \nu_i \epsilon_{\Theta\Theta}\right), \quad (8.33)$$

$$S_{(0)\Theta\Theta i} \equiv \varepsilon^2 \Sigma_i \widehat{S}_{(0)\Theta\Theta i} = S_i + \frac{E_i}{1-\nu_i^2}\left(\epsilon_{\Theta\Theta} + \nu_i \epsilon_{RR}\right), \quad (8.34)$$

where

$$\epsilon_{R\Theta} \equiv \varepsilon^2 \widehat{\epsilon}_{R\Theta} = \frac{1}{2}\left[v_{,R} + \frac{u_{,\Theta} - v}{R} + \frac{\Gamma_{,R} w_{,\Theta}}{R} + \frac{w_{,R} w_{,\Theta}}{R} - 2Z\left(\frac{w_{,R\Theta}}{R} - \frac{w_{,\Theta}}{R^2}\right)\right] \equiv \epsilon^0_{R\Theta} - Z\,k_{R\Theta}, \quad (8.35)$$

$$\epsilon_{RR} \equiv \varepsilon^2 \widehat{\epsilon}_{RR} = u_{,R} + \Gamma_{,R} w_{,R} + \frac{1}{2} w_{,R}^2 - Z\,w_{,RR} \equiv \epsilon^0_{RR} - Z\,k_{RR}, \quad (8.36)$$

$$\epsilon_{\Theta\Theta} \equiv \varepsilon^2 \widehat{\epsilon}_{\Theta\Theta} = \frac{v_{,\Theta} + u}{R} + \frac{w_{,\Theta}^2}{2R^2} - Z\left(\frac{w_{,R}}{R} + \frac{w_{,\Theta\Theta}}{R^2}\right) \equiv \epsilon^0_{\Theta\Theta} - Z\,k_{\Theta\Theta}, \quad (8.37)$$

the Z-independent terms of the last three equations are given by

$$\epsilon^0_{R\Theta} \equiv \frac{1}{2}\left(v_{,R} + \frac{u_{,\Theta} - v}{R} + \frac{\Gamma_{,R} w_{,\Theta}}{R} + \frac{w_{,R} w_{,\Theta}}{R}\right), \qquad k_{R\Theta} \equiv -\frac{w_{,R\Theta}}{R} + \frac{w_{,\Theta}}{R^2}, \quad (8.38)$$

$$\epsilon^0_{RR} \equiv u_{,R} + \Gamma_{,R} w_{,R} + \frac{1}{2} w_{,R}^2, \qquad k_{RR} \equiv w_{,RR}, \quad (8.39)$$

$$\epsilon^0_{\Theta\Theta} \equiv \frac{v_{,\Theta} + u}{R} + \frac{w_{,\Theta}^2}{2R^2}, \qquad k_{\Theta\Theta} \equiv \frac{w_{,R}}{R} + \frac{w_{,\Theta\Theta}}{R^2}, \quad (8.40)$$

and in (8.33) and (8.34) we have introduced the in-plane residual stresses defined by

$$S_i \equiv \varepsilon^2\,\Sigma_i\,\widehat{S}_i = \frac{1-2\nu_i}{1-\nu_i}\varepsilon^2 \Sigma_i \widehat{S}_i^{nm} = \frac{1-2\nu_i}{1-\nu_i} S_i^{nm} = -\frac{E_i}{1-\nu_i}\epsilon^{nm}, \quad (8.41)$$

where the last equality of (8.41) follows from (5.2), and refers to the particular case where the residual stresses are purely thermal in origin.

Note that only three in-plane constitutive equations (8.32), (8.33), and (8.34) are obtained as leading order results, from which the associated in-plane stress components can be determined in terms of the displacement components. The other constitutive relations provide the leading order (Kirchhoff-Love) forms (8.31) of the displacement components, but the associated out-of-plane stress components will be shown later to be determined in terms of the *in-plane stresses* via first integrals of the equilibrium equations.

8.2 Equilibrium Equations to Leading Order

Under the scalings of this Section, the equilibrium equations (7.38)–(7.40) can be written as

$$
\begin{aligned}
&\varepsilon^2\,\widehat{S}_{RRi,\rho} + \varepsilon^4\left(\widehat{U}_{,\rho}\widehat{S}_{RRi} + \frac{\widehat{U}_{,\Theta} - \widehat{V}}{\rho}\widehat{S}_{R\Theta i}\right)_{,\rho} + \varepsilon^4\left(\widehat{U}_{,\varsigma}\widehat{S}_{RZi}\right)_{\rho} \\
&+ \frac{1}{\rho}\left[\varepsilon^2\,\widehat{S}_{R\Theta i,\Theta} + \varepsilon^4\left(\widehat{U}_{,\rho}\widehat{S}_{R\Theta i} + \frac{\widehat{U}_{,\Theta} - \widehat{V}}{\rho}\widehat{S}_{\Theta\Theta i}\right)_{,\Theta} + \varepsilon^4\left(\widehat{U}_{,\varsigma}\widehat{S}_{\Theta Zi}\right)_{,\Theta}\right] \\
&+ \varepsilon^2\,\widehat{S}_{RZi,\varsigma} + \varepsilon^4\left(\widehat{U}_{,\rho}\widehat{S}_{RZi} + \frac{\widehat{U}_{,\Theta} - \widehat{V}}{\rho}\widehat{S}_{\Theta Zi}\right)_{,\varsigma} + \varepsilon^4\left(\widehat{U}_{,\varsigma}\widehat{S}_{ZZi}\right)_{,\varsigma} \\
&+ \frac{1}{\rho}\left\{\varepsilon^2\,\widehat{S}_{RRi} + \varepsilon^4\left(\widehat{U}_{,\rho}\widehat{S}_{RRi} + \frac{\widehat{U}_{,\Theta} - \widehat{V}}{\rho}\widehat{S}_{R\Theta i}\right) + \varepsilon^4\,\widehat{U}_{,\varsigma}\widehat{S}_{RZi}\right. \\
&\left. - \left[\varepsilon^2\,\widehat{S}_{\Theta\Theta i} + \varepsilon^4\left(\widehat{V}_{,\rho}\widehat{S}_{R\Theta i} + \frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho}\widehat{S}_{\Theta\Theta i}\right) + \varepsilon^4\,\widehat{V}_{,\varsigma}\widehat{S}_{\Theta Zi}\right]\right\} = 0,
\end{aligned}
\tag{8.42}
$$

$$
\begin{aligned}
&\varepsilon^2\,\widehat{S}_{R\Theta i,\rho} + \varepsilon^4\left(\widehat{V}_{,\rho}\widehat{S}_{RRi} + \frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho}\widehat{S}_{R\Theta i}\right)_{,\rho} + \varepsilon^4\left(\widehat{V}_{,\varsigma}\widehat{S}_{RZi}\right)_{\rho} \\
&+ \frac{1}{\rho}\left[\varepsilon^2\,\widehat{S}_{\Theta\Theta i,\Theta} + \varepsilon^4\left(\widehat{V}_{,\rho}\widehat{S}_{R\Theta i} + \frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho}\widehat{S}_{\Theta\Theta i}\right)_{,\Theta} + \varepsilon^4\left(\widehat{V}_{,\varsigma}\widehat{S}_{\Theta Zi}\right)_{,\Theta}\right] \\
&+ \varepsilon^2\,\widehat{S}_{\Theta Zi,\varsigma} + \varepsilon^4\left(\widehat{V}_{,\rho}\widehat{S}_{RZi} + \frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho}\widehat{S}_{\Theta Zi}\right)_{,\varsigma} + \varepsilon^4\left(\widehat{V}_{,\varsigma}\widehat{S}_{ZZi}\right)_{,\varsigma} \\
&+ \frac{1}{\rho}\left\{\varepsilon^2\,\widehat{S}_{R\Theta i} + \varepsilon^4\left(\widehat{V}_{,\rho}\widehat{S}_{RRi} + \frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho}\widehat{S}_{R\Theta i}\right) + \varepsilon^4\,\widehat{V}_{,\varsigma}\widehat{S}_{RZi}\right. \\
&\left. + \left[\varepsilon^2\,\widehat{S}_{R\Theta i} + \varepsilon^4\left(\widehat{U}_{,\rho}\widehat{S}_{R\Theta i} + \frac{\widehat{U}_{,\Theta} - \widehat{V}}{\rho}\widehat{S}_{\Theta\Theta i}\right) + \varepsilon^4\,\widehat{U}_{,\varsigma}\widehat{S}_{\Theta Zi}\right]\right\} = 0,
\end{aligned}
\tag{8.43}
$$

$$
\begin{aligned}
&\varepsilon^3\,\widehat{S}_{RZi,\rho} + \varepsilon^3\left(\widehat{\Gamma}_{,\rho}\widehat{S}_{RRi}\right)_{,\rho} + \varepsilon^3\left(\widehat{W}_{,\rho}\widehat{S}_{RRi} + \frac{\widehat{W}_{,\Theta}}{\rho}\widehat{S}_{R\Theta i}\right)_{,\rho} + \varepsilon^3\left(\widehat{W}_{,\varsigma}\widehat{S}_{RZi}\right)_{,\rho} \\
&+ \frac{1}{\rho}\left[\varepsilon^3\,\widehat{S}_{\Theta Zi,\Theta} + \varepsilon^3\left(\widehat{\Gamma}_{,\rho}\widehat{S}_{R\Theta i}\right)_{,\Theta} + \varepsilon^3\left(\widehat{W}_{,\rho}\widehat{S}_{R\Theta i} + \frac{\widehat{W}_{,\Theta}}{\rho}\widehat{S}_{\Theta\Theta i}\right)_{,\Theta} + \varepsilon^3\left(\widehat{W}_{,\varsigma}\widehat{S}_{\Theta Zi}\right)_{,\Theta}\right] \\
&+ \varepsilon^3\,\widehat{S}_{ZZi,\varsigma} + \varepsilon^3\left(\widehat{\Gamma}_{,\rho}\widehat{S}_{RZi}\right)_{,\varsigma} + \varepsilon^3\left(\widehat{W}_{,\rho}\widehat{S}_{RZi} + \frac{\widehat{W}_{,\Theta}}{\rho}\widehat{S}_{\Theta Zi}\right)_{,\varsigma} + \varepsilon^3\left(\widehat{W}_{,\varsigma}\widehat{S}_{ZZi}\right)_{,\varsigma} \\
&+ \frac{1}{\rho}\left[\varepsilon^3\,\widehat{S}_{RZi} + \varepsilon^3\,\widehat{\Gamma}_{,\rho}\widehat{S}_{RRi} + \varepsilon^3\left(\widehat{W}_{,\rho}\widehat{S}_{RRi} + \frac{\widehat{W}_{,\Theta}}{\rho}\widehat{S}_{R\Theta i}\right) + \varepsilon^3\,\widehat{W}_{,\varsigma}\widehat{S}_{RZi}\right] + \varepsilon^3\,\widehat{g} = 0.
\end{aligned}
\tag{8.44}
$$

To leading order, using the earlier result $\widehat{W}_{(0),\varsigma} = \widehat{w}_{,\varsigma} = 0$ in the third equation, we thus obtain

$$
\widehat{S}_{(0)RRi,\rho} + \frac{1}{\rho}\widehat{S}_{(0)R\Theta i,\Theta} + \widehat{S}_{(0)RZi,\varsigma} + \frac{1}{\rho}\left(\widehat{S}_{(0)RRi} - \widehat{S}_{(0)\Theta\Theta i}\right) = 0, \tag{8.45}
$$

$$\widehat{S}_{(0)R\Theta i,\rho} + \frac{1}{\rho}\widehat{S}_{(0)\Theta\Theta i,\Theta} + \widehat{S}_{(0)\Theta Zi,\zeta} + \frac{2}{\rho}\widehat{S}_{(0)R\Theta i} = 0, \tag{8.46}$$

$$\begin{aligned}&\left[\widehat{\omega}_{,\rho}\widehat{S}_{(0)RRi} + \frac{\widehat{w}_{,\Theta}}{\rho}\widehat{S}_{(0)R\Theta i} + \widehat{S}_{(0)RZi}\right]_{,\rho} + \frac{1}{\rho}\left[\widehat{\omega}_{,\rho}\widehat{S}_{(0)R\Theta i} + \frac{\widehat{w}_{,\Theta}}{\rho}\widehat{S}_{(0)\Theta\Theta i} + \widehat{S}_{(0)\Theta Zi}\right]_{,\Theta}\\ &+ \left[\widehat{\omega}_{,\rho}\widehat{S}_{(0)RZi} + \frac{\widehat{w}_{,\Theta}}{\rho}\widehat{S}_{(0)\Theta Zi} + \widehat{S}_{(0)ZZi}\right]_{,\zeta} + \frac{1}{\rho}\left[\widehat{\omega}_{,\rho}\widehat{S}_{(0)RRi} + \frac{\widehat{w}_{\Theta}}{\rho}\widehat{S}_{(0)R\Theta i} + \widehat{S}_{(0)RZi}\right] + \widehat{g} = 0,\end{aligned} \tag{8.47}$$

where we have introduced in the last equation a new ρ and Θ-dependent function $\widehat{\omega}$ defined by

$$\widehat{\omega}(\rho,\Theta) \equiv \widehat{w}(\rho,\Theta) + \widehat{\Gamma}(\rho). \tag{8.48}$$

8.3 Scaled Boundary Conditions of Pressure

The boundary conditions of pressure, equations (7.43)–(7.45), are given under the present scalings by

$$\begin{aligned}&\varepsilon^3\,\widehat{S}^{\pm}_{RZ} + \varepsilon^5\left[\widehat{U}^{\pm}_{,\rho}\widehat{S}^{\pm}_{RZ} + \left(\frac{\widehat{U}^{\pm}_{,\Theta} - \widehat{V}^{\pm}}{\rho}\right)\widehat{S}^{\pm}_{\Theta Z}\right] + \varepsilon^5\,\widehat{U}^{\pm}_{,\zeta}\widehat{S}^{\pm}_{ZZ}\\ &\quad = \varepsilon^5\,\widehat{p}^{\pm}\,\widehat{W}^{\pm}_{,\rho} - \varepsilon^7\left[\frac{\widehat{W}^{\pm}_{,\Theta}\widehat{V}^{\pm}_{,\rho} - \widehat{W}^{\pm}_{,\rho}\left(\widehat{V}^{\pm}_{,\Theta} + \widehat{U}^{\pm}\right)}{\rho^{\pm}}\right]\widehat{p}^{\pm},\end{aligned} \tag{8.49}$$

$$\begin{aligned}&\varepsilon^3\,\widehat{S}^{\pm}_{\Theta Z} + \varepsilon^5\left[\widehat{V}^{\pm}_{,\rho}\widehat{S}^{\pm}_{RZ} + \left(\frac{\widehat{V}^{\pm}_{,\Theta} + \widehat{U}^{\pm}}{\rho^{\pm}}\right)\widehat{S}^{\pm}_{\Theta Z}\right] + \varepsilon^5\,\widehat{V}^{\pm}_{,\zeta}\widehat{S}^{\pm}_{ZZ}\\ &\quad = \varepsilon^5\widehat{p}^{\pm}\frac{\widehat{W}^{\pm}_{,\Theta}}{\rho^{\pm}} + \varepsilon^7\left[\frac{\widehat{W}^{\pm}_{,\Theta}\widehat{U}^{\pm}_{,\rho} - \widehat{W}^{\pm}_{,\rho}\left(\widehat{U}^{\pm}_{,\Theta} - \widehat{V}^{\pm}\right)}{\rho^{\pm}}\right]\widehat{p}^{\pm},\end{aligned} \tag{8.50}$$

$$\begin{aligned}&\varepsilon^4\,\widehat{S}^{\pm}_{ZZ} + \varepsilon^4\left(\widehat{W}^{\pm}_{,\rho}\widehat{S}^{\pm}_{RZ} + \frac{\widehat{W}^{\pm}_{,\Theta}}{\rho^{\pm}}\widehat{S}^{\pm}_{\Theta Z}\right) + \varepsilon^4\,\widehat{W}^{\pm}_{,\zeta}\widehat{S}^{\pm}_{ZZ}\\ &= -\,\varepsilon^4\,\widehat{p}^{\pm} - \varepsilon^6\left(\widehat{U}^{\pm}_{,\rho} + \frac{\widehat{V}^{\pm}_{,\Theta} + \widehat{U}^{\pm}}{\rho^{\pm}}\right)\widehat{p}^{\pm} - \varepsilon^8\left[\widehat{U}^{\pm}_{,\rho}\left(\frac{\widehat{V}^{\pm}_{,\Theta} + \widehat{U}^{\pm}}{\rho^{\pm}}\right) - \widehat{V}^{\pm}_{,\rho}\left(\frac{\widehat{U}^{\pm}_{,\Theta} - \widehat{V}^{\pm}}{\rho^{\pm}}\right)\right]\widehat{p}^{\pm}.\end{aligned} \tag{8.51}$$

To leading order, they reduce to

$$\widehat{S}^{\pm}_{(0)RZ} = 0, \qquad \widehat{S}^{\pm}_{(0)\Theta Z} = , \qquad \widehat{S}^{\pm}_{(0)ZZ} = -\widehat{p}^{\pm}, \tag{8.52}$$

where the derivation of the last result for $\widehat{S}^{\pm}_{(0)ZZ}$ required the previous two results for $\widehat{S}^{\pm}_{(0)RZ}$ and $\widehat{S}^{\pm}_{(0)\Theta Z}$, and the fact that $\widehat{W}^{\pm}_{(0),\zeta} = 0$. Stated in terms of physical functions of the physical coordinates, these take the forms

$$S^{\pm}_{(0)RZ} = 0, \qquad S^{\pm}_{(0)\Theta Z} = 0, \qquad S^{\pm}_{(0)ZZ} \equiv \varepsilon^4\,\Sigma^{\pm}\,\widehat{S}^{\pm}_{(0)ZZ} = -\varepsilon^4\,\Sigma^{\pm}\,\widehat{p}^{\pm} \equiv -p^{\pm}. \tag{8.53}$$

8.4 Leading Order Equilibrium Solutions for the Out-of-Plane Stress Components

In terms of leading order physical functions of the physical coordinates R, Θ, and Z, the leading order equilibrium equations (8.45)–(8.47) are given by:

$$S_{(0)RZi,Z} + S_{(0)RRi,R} + \frac{1}{R}\left(S_{(0)R\Theta i,\Theta} + S_{(0)RRi} - S_{(0)\Theta\Theta i}\right) = 0, \tag{8.54}$$

$$S_{(0)\Theta Zi,Z} + S_{(0)R\Theta i,R} + \frac{1}{R}\left(S_{(0)\Theta\Theta i,\Theta} + 2\,S_{(0)R\Theta i}\right) = 0, \tag{8.55}$$

$$\begin{aligned}&\left[\omega_{,R}S_{(0)RZi} + \frac{w_{,\Theta}}{R}S_{(0)\Theta Zi} + S_{(0)ZZi}\right]_{,Z} + \frac{1}{R}\left[\omega_{,R}S_{(0)R\Theta i} + \frac{w_{,\Theta}}{R}S_{(0)\Theta\Theta i} + S_{(0)\Theta Zi}\right]_{,\Theta}\\ &+ \left[\omega_{,R}S_{(0)RRi} + \frac{w_{,\Theta}}{R}S_{(0)R\Theta i} + S_{(0)RZi}\right]_{,R} + \frac{1}{R}\left[\omega_{,R}S_{(0)RRi} + \frac{w_{,\Theta}}{R}S_{(0)R\Theta i} + S_{(0)RZi}\right] + \rho_{0i}\,g = 0,\end{aligned} \tag{8.56}$$

where

$$\omega(R,\Theta) \equiv \varepsilon\,a\,\widehat{\omega}(\rho,\Theta) \equiv w(R,\Theta) + \Gamma(R). \tag{8.57}$$

Equations (8.54)-(8.56) can be rewritten as

$$S_{(0)RZi,Z} + \frac{1}{R}\left[\left(RS_{(0)RRi}\right)_{,R} - S_{(0)\Theta\Theta i} + S_{(0)R\Theta i,\Theta}\right] = 0, \tag{8.58}$$

$$S_{(0)\Theta Zi,Z} + \frac{1}{R^2}\left(R^2\,S_{(0)R\Theta i}\right)_{,R} + \frac{1}{R}S_{(0)\Theta\Theta i,\Theta} = 0. \tag{8.59}$$

and

$$\begin{aligned}&\left[\omega_{,R}S_{(0)RZi}+\frac{w_{,\Theta}}{R}S_{(0)\Theta Zi} + S_{(0)ZZi}\right]_{,Z} + \frac{1}{R}\left[\omega_{,R}S_{(0)R\Theta i} + \frac{w_{,\Theta}}{R}S_{(0)\Theta\Theta i} + S_{(0)\Theta Zi}\right]_{,\Theta}\\ &\quad + \frac{1}{R}\left(R\,S_{(0)RZi}\right)_{,R} + \frac{1}{R}\left[R\left(\omega_{,R}S_{(0)RRi} + \frac{w_{,\Theta}}{R}S_{(0)R\Theta i}\right)\right]_{,R} + \rho_{0i}\,g = 0,\end{aligned} \tag{8.60}$$

respectively. Now, according to (8.32)–(8.40), $S_{(0)R\Theta i}$, $S_{(0)RRi}$, and $S_{(0)\Theta\Theta i}$ are linear functions of Z, with coefficients depending only on R and Θ, hence equations (8.58) and (8.59) can be written as

$$S_{(0)RZi,Z} + a_{0i}(R,\Theta) + Z\,a_{1i}(R,\Theta) = 0, \qquad S_{(0)\Theta Zi,Z} + b_{0i}(R,\Theta) + Z\,b_{1i}(R,\Theta) = 0, \tag{8.61}$$

respectively, where a_{0i}, a_{1i}, b_{0i}, and b_{1i} are rather complicated Z-independent functions. These two equations can be easily solved to obtain

$$S_{(0)RZi} + Z\,a_{0i} + \frac{Z^2}{2}\,a_{1i} = F_{RZi}, \qquad S_{(0)\Theta Zi} + Z\,b_{0i} + \frac{Z^2}{2}\,b_{1i} = F_{\Theta Zi}, \tag{8.62}$$

where F_{RZi} and $F_{\Theta Zi}$ are arbitrary functions of R and Θ only. Applying the first two boundary conditions of (8.53), we obtain from the first and second equations of (8.62):

$$F_{RZs} = \frac{h}{2}\,a_{0s} + \frac{h^2}{8}\,a_{1s}, \qquad F_{RZc} = -\frac{h}{2}\,a_{0c} + \frac{h^2}{8}\,a_{1c}, \tag{8.63}$$

and

$$F_{\Theta Zs} = \frac{h}{2}\,b_{0s} + \frac{h^2}{8}\,b_{1s}, \qquad F_{\Theta Zc} = -\frac{h}{2}\,b_{0c} + \frac{h^2}{8}\,b_{1c}, \tag{8.64}$$

respectively. For a two-layer laminate, the solutions for the stress components in the two materials can thus be written as

$$S_{(0)RZs} = \left(\frac{h}{2} - Z\right)\,a_{0s} + \frac{1}{8}\left(h^2 - 4Z^2\right)\,a_{1s}, \qquad S_{(0)RZc} = -\left(\frac{h}{2} + Z\right)\,a_{0c} + \frac{1}{8}\left(h^2 - 4Z^2\right)\,a_{1c}, \tag{8.65}$$

and

$$S_{(0)\Theta Zs} = \left(\frac{h}{2} - Z\right)\,b_{0s} + \frac{1}{8}\left(h^2 - 4Z^2\right)\,b_{1s}, \qquad S_{(0)\Theta Zc} = -\left(\frac{h}{2} + Z\right)\,b_{0c} + \frac{1}{8}\left(h^2 - 4Z^2\right)\,b_{1c}, \tag{8.66}$$

respectively, where $i = 1 = c$ denotes the coating and $i = 2 = s$ denotes the membrane substrate.

Next, we note that since $S_{(0)RZi}$ and $S_{(0)\Theta Zi}$ are quadratic in Z, the axial equilibrium equation (8.60) can be written as

$$\left[\omega_{,R} S_{(0)RZi} + \frac{w_{,\Theta}}{R} S_{(0)\Theta Zi} + S_{(0)ZZi}\right]_{,Z} + c_{0i}(R,\Theta) + Z\, c_{1i}(R,\Theta) + Z^2\, c_{2i}(R,\Theta) = 0, \tag{8.67}$$

where c_{0i}, c_{1i}, and c_{2i} are complicated functions of R and Θ only (note that c_{0i} includes the term $\rho_{0i}\, g$). The general solution of (8.67) is

$$\omega_{,R} S_{(0)RZi} + \frac{w_{,\Theta}}{R} S_{(0)\Theta Zi} + S_{(0)ZZi} + Z\, c_{0i}(R,\Theta) + \frac{Z^2}{2} c_{1i}(R,\Theta) + \frac{Z^3}{3} c_{2i}(R,\Theta) = F_{ZZi}(R,\Theta), \tag{8.68}$$

where F_{ZZi} is another arbitrary function of R and Θ only. Applying the three boundary conditions of (8.53), we obtain

$$F_{ZZs} = -p^+ + \frac{h}{2} c_{0s} + \frac{h^2}{8} c_{1s} + \frac{h^3}{24} c_{2s}, \text{ and } F_{ZZc} = -p^- - \frac{h}{2} c_{0c} + \frac{h^2}{8} c_{1c} - \frac{h^3}{24} c_{2c}. \tag{8.69}$$

The solutions for this stress component are thus

$$\begin{aligned} S_{(0)ZZs} = -p^+ + \left(\frac{h}{2} - Z\right) c_{0s} + \frac{1}{8}\left(h^2 - 4Z^2\right) c_{1s} + \frac{1}{24}\left(h^3 - 8Z^3\right) c_{2s} \\ - \omega_{,R} S_{(0)RZs} - \frac{w_{,\Theta}}{R} S_{(0)\Theta Zs} \end{aligned} \tag{8.70}$$

in the membrane substrate, and

$$\begin{aligned} S_{(0)ZZc} = -p^- - \left(\frac{h}{2} + Z\right) c_{0c} + \frac{1}{8}\left(h^2 - 4Z^2\right) c_{1c} - \frac{1}{24}\left(h^3 + 8Z^3\right) c_{2c} \\ - \omega_{,R} S_{(0)RZc} - \frac{w_{,\Theta}}{R} S_{(0)\Theta Zc} \end{aligned} \tag{8.71}$$

in the coating.

8.5 Continuity Conditions on the Out-of-Plane Stress Components

Requiring that the stress components $S_{(0)RZ}$ and $S_{(0)\Theta Zs}$ be continuous at the interface $Z_I = (h_c - h_s)/2$, we find from (8.65) and (8.66) that we must have

$$S_{(0)RZs}(Z_I) - S_{(0)RZc}(Z_I) = h_s\, a_{0s} + h_c\, a_{0c} + \frac{1}{2} h_c\, h_s\, (a_{1s} - a_{1c}) = 0, \tag{8.72}$$

and

$$S_{(0)\Theta Zs}(Z_I) - S_{(0)\Theta Zc}(Z_I) = h_s\, b_{0s} + h_c\, b_{0c} + \frac{1}{2} h_c\, h_s\, (b_{1s} - b_{1c}) = 0, \tag{8.73}$$

respectively. Assuming these continuity conditions to hold, the continuity condition for $S_{(0)ZZ}$ at the interface follows from equations (8.70) and (8.71), viz.,

$$\begin{aligned} S_{(0)ZZs}(Z_I) - S_{(0)ZZc}(Z_I) = p + h_s\, c_{0s} + h_c\, c_{0c} + \frac{1}{2} h_c\, h_s\, (c_{1s} - c_{1c}) \\ + \frac{1}{12} h_s \left(3h_c^2 + h_s^2\right) c_{2s} + \frac{1}{12} h_c \left(3h_s^2 + h_c^2\right) c_{2c} = 0, \end{aligned} \tag{8.74}$$

where

$$p \equiv p^- - p^+ \tag{8.75}$$

is the pressure *difference* between the lower and upper faces of the deformed configuration. Equations (8.72)-(8.74) represent conditions that must be satisfied by the fourteen Z-independent functions a_{0i}, a_{1i}, b_{0i}, b_{1i}, c_{0i}, c_{1i}, and c_{2i} ($i = c$ or s), in order for the out-of-plane stress components to be continuous across the coating/membrane interface.

To facilitate the development of the continuity conditions, we first separate each of the in-plane constitutive relations (8.32)-(8.34) into its Z-dependent and Z-independent parts:

$$S_{(0)R\Theta i} = \sigma_{R\Theta i} - Z\eta_{R\Theta i}, \qquad S_{(0)RRi} = \sigma_{Ri} - Z\eta_{Ri}, \qquad S_{(0)\Theta\Theta i} = \sigma_{\Theta i} - Z\eta_{\Theta i}, \tag{8.76}$$

where

$$\sigma_{R\Theta i} \equiv G_i\,\epsilon^0_{R\Theta}, \qquad \eta_{R\Theta i} \equiv G_i\,k_{R\Theta}, \tag{8.77}$$

$$\sigma_{Ri} \equiv S_i + Q_i\left(\epsilon^0_{RR} + \nu_i\epsilon^0_{\Theta\Theta}\right), \qquad \eta_{Ri} \equiv Q_i\left(k_{RR} + \nu_i k_{\Theta\Theta}\right), \tag{8.78}$$

$$\sigma_{\Theta i} \equiv S_i + Q_i\left(\epsilon^0_{\Theta\Theta} + \nu_i\epsilon^0_{RR}\right), \qquad \eta_{\Theta i} \equiv Q_i\left(k_{\Theta\Theta} + \nu_i k_{RR}\right). \tag{8.79}$$

In (8.78) and (8.79) we have introduced material parameters Q_i, defined in terms of the moduli and Poisson's ratios by

$$Q_s = \frac{E_s}{1-\nu_s^2}, \qquad Q_c = \frac{E_c}{1-\nu_c^2}, \tag{8.80}$$

and from (8.38)–(8.40), the Z-independent "strains" and "curvatures" are given in terms of the leading order displacement components by

$$\epsilon^0_{R\Theta} \equiv \frac{1}{2}\left(v_{,R} + \frac{u_{,\Theta} - v}{R} + \frac{\Gamma_{,R}w_{,\Theta}}{R} + \frac{w_{,R}w_{,\Theta}}{R}\right), \qquad k_{R\Theta} \equiv -\frac{w_{,R\Theta}}{R} + \frac{w_{,\Theta}}{R^2}, \tag{8.81}$$

$$\epsilon^0_{RR} \equiv u_{,R} + \Gamma_{,R}w_{,R} + \frac{1}{2}w_{,R}^2, \qquad k_{RR} \equiv w_{,RR}, \tag{8.82}$$

$$\epsilon^0_{\Theta\Theta} \equiv \frac{v_{,\Theta} + u}{R} + \frac{w_{,\Theta}^2}{2R^2}, \qquad k_{\Theta\Theta} \equiv \frac{w_{,R}}{R} + \frac{w_{,\Theta\Theta}}{R^2}. \tag{8.83}$$

The development of the continuity equations first requires identification of the fourteen Z-independent functions $a_{0i}, a_{1i}, b_{0i}, b_{1i}, c_{0i}, c_{1i}$, and c_{2i} ($i = c$ or s). Comparison of (8.62) with equations (8.58) and (8.59) yields

$$a_{0i} + Z\,a_{1i} = \frac{1}{R}\left[\left(RS_{(0)RRi}\right)_{,R} - S_{(0)\Theta\Theta i} + S_{(0)R\Theta i,\Theta}\right], \tag{8.84}$$

$$b_{0i} + Z\,b_{1i} = \frac{1}{R^2}\left(R^2\,S_{(0)R\Theta i}\right)_{,R} + \frac{1}{R}S_{(0)\Theta\Theta i,\Theta}. \tag{8.85}$$

Similarly, comparison of (8.67) with (8.60) yields

$$c_{0i} + Z\,c_{1i} + Z^2\,c_{2i} = \frac{1}{R}\left[\omega_{,R}S_{(0)R\Theta i} + \frac{w_{,\Theta}}{R}S_{(0)\Theta\Theta i} + S_{(0)\Theta Zi}\right]_{,\Theta} + \frac{1}{R}\left(R\,S_{(0)RZi}\right)_{,R} + \frac{1}{R}\left[R\left(\omega_{,R}S_{(0)RRi} + \frac{w_{,\Theta}}{R}S_{(0)R\Theta i}\right)\right]_{,R} + \rho_{0i}\,g. \tag{8.86}$$

Substituting from equations (8.76) in the right-hand sides of (8.84) and (8.85), we easily find

$$a_{0i} = \frac{1}{R}\left[\left(R\sigma_{Ri}\right)_{,R} - \sigma_{\Theta i} + \sigma_{R\Theta i,\Theta}\right], \qquad a_{1i} = -\frac{1}{R}\left[\left(R\eta_{Ri}\right)_{,R} - \eta_{\Theta i} + \eta_{R\Theta i,\Theta}\right], \tag{8.87}$$

$$b_{0i} = \frac{1}{R^2}(R^2\sigma_{R\Theta i})_{,R} + \frac{1}{R}\sigma_{\Theta i,\Theta}, \qquad b_{1i} = -\frac{1}{R^2}(R^2\eta_{R\Theta i})_{,R} - \frac{1}{R}\eta_{\Theta i,\Theta}, \tag{8.88}$$

for either $i = c$ or s. Substituting from equations (8.76), (8.65), and (8.66) in the right-hand side of (8.86), we eventually obtain for $i = s$:

$$c_{0s} = \rho_{0s}\,g + \frac{1}{R}\Big[\left(\omega_{,R}\sigma_{R\Theta s} + \frac{w_{,\Theta}}{R}\sigma_{\Theta s} + \frac{h}{2}b_{0s} + \frac{h^2}{8}b_{1s}\right)_{,\Theta} \tag{8.89}$$

$$+\left(\frac{h}{2}Ra_{0s} + \frac{h^2}{8}Ra_{1s} + R\omega_{,R}\sigma_{Rs} + \frac{w_{,\Theta}}{R}\sigma_{R\Theta s}\right)_{,R}\Big], \tag{8.90}$$

$$c_{1s} = -\frac{1}{R}\left[\left(\omega_{,R}\eta_{R\Theta s} + \frac{w_{,\Theta}}{R}\eta_{\Theta s} + b_{0s}\right)_{,\Theta} + \left(Ra_{0s} + R\omega_{,R}\eta_{Rs} + \frac{w_{,\Theta}}{R}\eta_{R\Theta s}\right)_{,R}\right], \tag{8.91}$$

$$c_{2s} = -\frac{1}{2R}\left[b_{1s,\Theta} + (Ra_{1s})_{,R}\right], \tag{8.92}$$

and for $i = c$:

$$c_{0c} = \rho_{0c}\,g + \frac{1}{R}\Big[\left(\omega_{,R}\sigma_{R\Theta c} + \frac{w_{,\Theta}}{R}\sigma_{\Theta c} - \frac{h}{2}b_{0c} + \frac{h^2}{8}b_{1c}\right)_{,\Theta} \tag{8.93}$$

$$+\left(-\frac{h}{2}Ra_{0c} + \frac{h^2}{8}Ra_{1c} + R\omega_{,R}\sigma_{Rc} + \frac{w_{,\Theta}}{R}\sigma_{R\Theta c}\right)_{,R}\Big], \tag{8.94}$$

$$c_{1c} = -\frac{1}{R}\left[\left(\omega_{,R}\eta_{R\Theta c} + \frac{w_{,\Theta}}{R}\eta_{\Theta c} + b_{0c}\right)_{,\Theta} + \left(Ra_{0c} + R\omega_{,R}\eta_{Rc} + \frac{w_{,\Theta}}{R}\eta_{R\Theta c}\right)_{,R}\right], \tag{8.95}$$

$$c_{2c} = -\frac{1}{2R}\left[b_{1c,\Theta} + (Ra_{1c})_{,R}\right]. \tag{8.96}$$

Beginning with the continuity condition for $S_{(0)RZ}$ given in (8.72), we substitute from (8.87) into that equation, then expand the derivatives and multiply the result through by R, to obtain

$$\begin{aligned} &h_s\,(R\sigma_{Rs,R} + \sigma_{Rs} - \sigma_{\Theta s} + \sigma_{R\Theta s,\Theta}) + h_c\,(R\sigma_{Rc,R} + \sigma_{Rc} - \sigma_{\Theta c} + \sigma_{R\Theta c,\Theta}) \\ &\quad - \frac{1}{2}h_c\,h_s\,[(R\eta_{Rs,R} + \eta_{Rs} - \eta_{\Theta s} + \eta_{R\Theta s,\Theta}) - (R\eta_{Rc,R} + \eta_{Rc} - \eta_{\Theta c} + \eta_{R\Theta c,\Theta})] = 0. \end{aligned}$$

Substituting from equations (8.77)–(8.79) into this equation yields, assuming the residual stresses S_c and S_s to be constants,

$$\begin{aligned} &R\,(A\,\epsilon^0_{RR,R} + A_\nu\,\epsilon^0_{\Theta\Theta,R}) + (A - A_\nu)\,(\epsilon^0_{RR} - \epsilon^0_{\Theta\Theta}) + A_\Theta\,\epsilon^0_{R\Theta,\Theta} \\ &\quad + R\,(B\,k_{RR,R} + B_\nu\,k_{\Theta\Theta,R}) + (B - B_\nu)\,(k_{RR} - k_{\Theta\Theta}) + B_\Theta\,k_{R\Theta,\Theta} = 0, \end{aligned} \tag{8.97}$$

where we have introduced the following constants linear and quadratic, respectively, in the thicknesses h_c and h_s:

$$A = h_s\,Q_s + h_c\,Q_c, \qquad A_\nu = h_sQ_s\nu_s + h_c\,Q_c\nu_c, \qquad A_\Theta = h_s\,G_s + h_c\,G_c, \tag{8.98}$$

$$B = \frac{1}{2}h_c\,h_s\,(Q_c - Q_s), \qquad B_\nu = \frac{1}{2}h_c\,h_s\,(Q_c\nu_c - Q_s\nu_s) \qquad B_\Theta = \frac{1}{2}h_c\,h_s\,(G_c - G_s). \tag{8.99}$$

Similarly, substituting from (8.88) into the continuity condition (8.73) for $S_{(0)\Theta Z}$, we obtain after some algebra the following form for the second continuity condition:

$$A_\Theta\,(R^2\epsilon^0_{R\Theta})_{,R} + R\,(A\,\epsilon^0_{\Theta\Theta} + A_\nu\,\epsilon^0_{RR})_{,\Theta} + B_\Theta\,(R^2k_{R\Theta})_{,R} + R\,(B\,k_{\Theta\Theta} + B_\nu\,k_{RR})_{,\Theta} = 0, \tag{8.100}$$

where the constants are defined in (8.98) and (8.99), and we again assumed the residual stresses to be constants.

The final continuity condition (8.74) involves some rather tedious algebra, which we omit here. However, in order to write the result in terms of constants previously defined in (8.98) and (8.99), we remark that we make use of the following identity:

$$h_s^2\,\chi_{0s} - h_c^2\,\chi_{0c} \equiv (h_s - h_c)\,(h_s\,\chi_{0s} + h_c\,\chi_{0c}) - h_c\,h_s\,(\chi_{0c} - \chi_{0s}), \tag{8.101}$$

where the Z-independent function χ may be either a or b. Using this identity, the final continuity condition can be brought to the form

$$\begin{aligned}
&\Big\{\frac{1}{2}\,(h_s - h_c)\,\big[R\,(A\,\epsilon^0_{RR,R} + A_\nu\,\epsilon^0_{\Theta\Theta,R}) + (A - A_\nu)\,(\epsilon^0_{RR} - \epsilon^0_{\Theta\Theta}) + A_\Theta\,\epsilon^0_{R\Theta,\Theta}\big]\\
&\qquad - \big[R\,(B\,\epsilon^0_{RR,R} + B_\nu\,\epsilon^0_{\Theta\Theta,R}) + (B - B_\nu)\,(\epsilon^0_{RR} - \epsilon^0_{\Theta\Theta}) + B_\Theta\,\epsilon^0_{R\Theta,\Theta}\big]\\
&\qquad - \big[R\,(\overline{D}\,k_{RR,R} + \overline{D}_\nu\,k_{\Theta\Theta,R}) + (\overline{D} - \overline{D}_\nu)\,(k_{RR} - k_{\Theta\Theta}) + \overline{D}_\Theta\,k_{R\Theta,\Theta}\big]\\
&\quad + R\Big[\omega_{,R}\,(\mathcal{N} + A\,\epsilon^0_{RR} + A_\nu\,\epsilon^0_{\Theta\Theta} + B\,k_{RR} + B_\nu\,k_{\Theta\Theta}) + \frac{w_{,\Theta}}{R}\,(A_\Theta\,\epsilon^0_{R\Theta} + B_\Theta\,k_{R\Theta})\Big]\Big\}_{,R}\\
&\quad + \frac{1}{R^2}\Big\{\frac{1}{2}\,(h_s - h_c)\,\Big[A_\Theta\,(R^2\epsilon^0_{R\Theta})_{,R} + R\,(A\epsilon^0_{\Theta\Theta,\Theta} + A_\nu\,\epsilon^0_{RR,\Theta})\Big] - \Big[B_\Theta\,(R^2\epsilon^0_{R\Theta})_{,R}\\
&\qquad\qquad + R\,(B\epsilon^0_{\Theta\Theta,\Theta} + B_\nu\,\epsilon^0_{RR,\Theta})\Big]\\
&\qquad - \Big[\overline{D}_\Theta\,(R^2 k_{R\Theta})_{,R} + R\,(\overline{D}k_{\Theta\Theta,\Theta} + \overline{D}_\nu\,k_{RR,\Theta})\Big] + R^2\,\omega_{,R}\,(A_\Theta\,\epsilon^0_{R\Theta} + B_\Theta\,k_{R\Theta})\\
&\qquad + R w_{,\Theta}\,(\mathcal{N} + A\,\epsilon^0_{\Theta\Theta} + A_\nu\,\epsilon^0_{RR} + B\,k_{\Theta\Theta} + B_\nu\,k_{RR})\Big\}_{,\Theta} + R\,(p + \gamma_0\,g) = 0,
\end{aligned} \tag{8.102}$$

where

$$\gamma_0 \equiv h_s\,\rho_{0s} + h_c\,\rho_{0c}, \tag{8.103}$$

is the *areal* density of the coated membrane (mass per unit area, Kg/m^2, of the circular disk perpendicular to the axis), and we have introduced three new constants, each cubic in the thickness h_s, defined by

$$\overline{D} = \frac{h_s^3}{12}[(1 + 3\mathcal{H})\,Q_s + (3 + \mathcal{H})\,\mathcal{H}^2\,Q_c], \tag{8.104}$$

$$\overline{D}_\nu = \frac{h_s^3}{12}[(1 + 3\mathcal{H})\,Q_s\nu_s + (3 + \mathcal{H})\,\mathcal{H}^2\,Q_c\nu_c], \tag{8.105}$$

$$\overline{D}_\Theta = \frac{h_s^3}{12}[(1 + 3\mathcal{H})\,G_s + (3 + \mathcal{H})\,\mathcal{H}^2\,G_c], \tag{8.106}$$

as well as a constant $\mathcal{N}$ having units of N/m:

$$\mathcal{N} = h_s\,S_s + h_c\,S_c. \tag{8.107}$$

In the definitions (8.104)–(8.106) we have eliminated h_c, introducing instead the thickness ratio $\mathcal{H}$ defined by

$$\mathcal{H} \equiv \frac{h_c}{h_s}. \tag{8.108}$$

The terms in (8.102) involving the constants A, A_ν, and A_Θ can be eliminated in favor of terms involving B, B_ν, and B_Θ and three new constants D, D_ν, and D_Θ, by using the first two continuity conditions (8.97)

and (8.100). With these manipulations the third continuity equation simplifies a bit to

$$\begin{aligned}
&\Big\{ - \left[R\left(B\,\epsilon^0_{RR,R} + B_\nu\,\epsilon^0_{\Theta\Theta,R}\right) + (B - B_\nu)\left(\epsilon^0_{RR} - \epsilon^0_{\Theta\Theta}\right) + B_\Theta\,\epsilon^0_{R\Theta,\Theta}\right] \\
&\qquad - \left[R\left(D\,k_{RR,R} + D_\nu\,k_{\Theta\Theta,R}\right) + (D - D_\nu)\left(k_{RR} - k_{\Theta\Theta}\right) + D_\Theta\,k_{R\Theta,\Theta}\right] \\
&\quad + R\left[\omega_{,R}\left(\mathcal{N} + A\,\epsilon^0_{RR} + A_\nu\,\epsilon^0_{\Theta\Theta} + B\,k_{RR} + B_\nu\,k_{\Theta\Theta}\right) + \frac{w_{,\Theta}}{R}\left(A_\Theta\,\epsilon^0_{R\Theta} + B_\Theta\,k_{R\Theta}\right)\right]\Big\}_{,R} \\
&\qquad + \frac{1}{R^2}\Big\{ - \left[B_\Theta\left(R^2\epsilon^0_{R\Theta}\right)_{,R} + R\left(B\epsilon^0_{\Theta\Theta,\Theta} + B_\nu\,\epsilon^0_{RR,\Theta}\right)\right] \\
&\quad - \left[D_\Theta\left(R^2 k_{R\Theta}\right)_{,R} + R\left(Dk_{\Theta\Theta,\Theta} + D_\nu\,k_{RR,\Theta}\right)\right] + R^2\,\omega_{,R}\left(A_\Theta\,\epsilon^0_{R\Theta} + B_\Theta\,k_{R\Theta}\right) \\
&\qquad + R\,w_{,\Theta}\left(\mathcal{N} + A\,\epsilon^0_{\Theta\Theta} + A_\nu\,\epsilon^0_{RR} + B\,k_{\Theta\Theta} + B_\nu\,k_{RR}\right)\Big\}_{,\Theta} + R\left(p + \gamma_0\,g\right) = 0, \qquad (8.109)
\end{aligned}$$

where D, D_ν, and D_Θ are defined by

$$D = \frac{h_s^3}{12}\left[(1 + 3\mathcal{H}^2)\,Q_s + (3 + \mathcal{H}^2)\,\mathcal{H}\,Q_c\right], \qquad (8.110)$$

$$D_\nu = \frac{h_s^3}{12}\left[(1 + 3\mathcal{H}^2)\,Q_s\nu_s + (3 + \mathcal{H}^2)\,\mathcal{H}\,Q_c\nu_c\right], \qquad (8.111)$$

$$D_\Theta = \frac{h_s^3}{12}\left[(1 + 3\mathcal{H}^2)\,G_s + (3 + \mathcal{H}^2)\,\mathcal{H}G_c\right]. \qquad (8.112)$$

We conclude this Section by summarizing the continuity conditions in terms of the strains and curvatures (and their partial derivatives):

$$\begin{aligned}
&R\left(A\,\epsilon^0_{RR,R} + A_\nu\,\epsilon^0_{\Theta\Theta,R}\right) + (A - A_\nu)\left(\epsilon^0_{RR} - \epsilon^0_{\Theta\Theta}\right) + A_\Theta\,\epsilon^0_{R\Theta,\Theta} \\
&\qquad + R\left(B\,k_{RR,R} + B_\nu\,k_{\Theta\Theta,R}\right) + (B - B_\nu)\left(k_{RR} - k_{\Theta\Theta}\right) + B_\Theta\,k_{R\Theta,\Theta} = 0, \qquad (8.113)
\end{aligned}$$

$$A_\Theta\left(R^2\epsilon^0_{R\Theta}\right)_{,R} + R\left(A\,\epsilon^0_{\Theta\Theta} + A_\nu\,\epsilon^0_{RR}\right)_{,\Theta} + B_\Theta\left(R^2 k_{R\Theta}\right)_{,R} + R\left(B\,k_{\Theta\Theta} + B_\nu\,k_{RR}\right)_{,\Theta} = 0, \qquad (8.114)$$

$$\begin{aligned}
&\Big\{ - \left[R\left(B\,\epsilon^0_{RR,R} + B_\nu\,\epsilon^0_{\Theta\Theta,R}\right) + (B - B_\nu)\left(\epsilon^0_{RR} - \epsilon^0_{\Theta\Theta}\right) + B_\Theta\,\epsilon^0_{R\Theta,\Theta}\right] \\
&\qquad - \left[R\left(D\,k_{RR,R} + D_\nu\,k_{\Theta\Theta,R}\right) + (D - D_\nu)\left(k_{RR} - k_{\Theta\Theta}\right) + D_\Theta\,k_{R\Theta,\Theta}\right] \\
&\quad + R\left[\left(w_{,R} + \Gamma_{,R}\right)\left(\mathcal{N} + A\,\epsilon^0_{RR} + A_\nu\,\epsilon^0_{\Theta\Theta} + B\,k_{RR} + B_\nu\,k_{\Theta\Theta}\right) + \frac{w_{,\Theta}}{R}\left(A_\Theta\,\epsilon^0_{R\Theta} + B_\Theta\,k_{R\Theta}\right)\right]\Big\}_{,R} \\
&\qquad + \frac{1}{R^2}\Big\{ - \left[B_\Theta\left(R^2\epsilon^0_{R\Theta}\right)_{,R} + R\left(B\epsilon^0_{\Theta\Theta,\Theta} + B_\nu\,\epsilon^0_{RR,\Theta}\right)\right] \\
&\quad - \left[D_\Theta\left(R^2 k_{R\Theta}\right)_{,R} + R\left(Dk_{\Theta\Theta,\Theta} + D_\nu\,k_{RR,\Theta}\right)\right] + R^2\left(w_{,R} + \Gamma_{,R}\right)\left(A_\Theta\,\epsilon^0_{R\Theta} + B_\Theta\,k_{R\Theta}\right) \\
&\qquad + R\,w_{,\Theta}\left(\mathcal{N} + A\,\epsilon^0_{\Theta\Theta} + A_\nu\,\epsilon^0_{RR} + B\,k_{\Theta\Theta} + B_\nu\,k_{RR}\right)\Big\}_{,\Theta} + R\left(p + \gamma_0\,g\right) = 0, \qquad (8.115)
\end{aligned}$$

where the definitions of the constants are repeated here, for convenience:

$$\mathcal{N} = h_s\,S_s + h_c\,S_c, \qquad (8.116)$$

$$A = h_s\,Q_s + h_c\,Q_c, \qquad A_\nu = h_s Q_s\nu_s + h_c\,Q_c\nu_c, \qquad A_\Theta = h_s\,G_s + h_c\,G_c, \qquad (8.117)$$

$$B = \frac{1}{2}h_c\,h_s\left(Q_c - Q_s\right), \qquad B_\nu = \frac{1}{2}h_c\,h_s\left(Q_c\nu_c - Q_s\nu_s\right), \qquad B_\Theta = \frac{1}{2}h_c\,h_s\left(G_c - G_s\right), \qquad (8.118)$$

$$D = \frac{h_s^3}{12}\left[(1 + 3\mathcal{H}^2)Q_s + (3 + \mathcal{H}^2)\mathcal{H}Q_c\right], \tag{8.119}$$

$$D_\nu = \frac{h_s^3}{12}\left[(1 + 3\mathcal{H}^2)Q_s\nu_s + (3 + \mathcal{H}^2)\mathcal{H}Q_c\nu_c\right], \tag{8.120}$$

$$D_\Theta = \frac{h_s^3}{12}\left[(1 + 3\mathcal{H}^2)G_s + (3 + \mathcal{H}^2)\mathcal{H}G_c\right]. \tag{8.121}$$

Note that in (8.115), we have replaced two occurrences of the partial derivative $\omega_{,R} \equiv w_{,R} + \Gamma_{,R}$. For comparisons with Wittrick's work [16], we remark that our ω corresponds to the function he denotes by w, our Γ corresponds to the function he denotes by w_0, and our w corresponds to the function he denotes by w'. The constants given above differ from those in equations (8) of [16] due to a difference in the choice of origin of coordinates (our origin is at the center of the *middle* plane of the reference placement, while Wittrick's corresponds to the center of the *interface* plane between coating and substrate of the reference placement). The constants A_Θ, B_Θ, and D_Θ do not appear in Wittrick's paper, since he treats only the axisymmetric problem.

8.6 Formulation of Equilibrium Equations in Terms of Stress Resultants and Stress Couples

It is more common to find the equilibrium equations for a shell or plate presented in terms of stress resultants and couples, and in many respects such a formulation is simpler than (though equivalent to) the one discussed in the previous two Subsections. However, it was felt that the approach via continuity conditions was sufficiently novel to include in this Report. Here, we derive equations involving stress resultants and stress couples from the fundamental equilibrium equations (8.58)–(8.60), which we repeat here for convenience:

$$S_{(0)RZi,Z} + \frac{1}{R}\left[\left(RS_{(0)RRi}\right)_{,R} - S_{(0)\Theta\Theta i} + S_{(0)R\Theta i,\Theta}\right] = 0, \tag{8.122}$$

$$S_{(0)\Theta Zi,Z} + \frac{1}{R^2}\left(R^2\, S_{(0)R\Theta i}\right)_{,R} + \frac{1}{R}S_{(0)\Theta\Theta i,\Theta} = 0. \tag{8.123}$$

$$\begin{aligned}\left[\omega_{,R}S_{(0)RZi}+\frac{w_{,\Theta}}{R}S_{(0)\Theta Zi} + S_{(0)ZZi}\right]_{,Z} + \frac{1}{R}\left[\omega_{,R}S_{(0)R\Theta i} + \frac{w_{,\Theta}}{R}S_{(0)\Theta\Theta i} + S_{(0)\Theta Zi}\right]_{,\Theta} \\ + \frac{1}{R}\left[R\left(\omega_{,R}S_{(0)RRi} + \frac{w_{,\Theta}}{R}S_{(0)R\Theta i} + S_{(0)RZi}\right)\right]_{,R} + \rho_{0i}\, g = 0,\end{aligned} \tag{8.124}$$

where the last equation is a slightly modified version of (8.60). Each of equations (8.122)–(8.124) is first integrated through the thickness to eliminate the Z-dependence. Introducing radial, circumferential, and in-plane shear stress resultants, defined by the following integrals through the thickness (refer to the lower portion of Figure 2):

$$N_R \equiv \int_{-h/2}^{h/2} S_{(0)RRi}\, dZ = \int_{-h/2}^{(h_c-h_s)/2} S_{(0)RRc}\, dZ + \int_{(h_c-h_s)/2}^{h/2} S_{(0)RRs}\, dZ, \tag{8.125}$$

$$N_{R\Theta} \equiv \int_{-h/2}^{h/2} S_{(0)R\Theta i}\, dZ = \int_{-h/2}^{(h_c-h_s)/2} S_{(0)R\Theta c}\, dZ + \int_{(h_c-h_s)/2}^{h/2} S_{(0)R\Theta s}\, dZ, \tag{8.126}$$

$$N_\Theta \equiv \int_{-h/2}^{h/2} S_{(0)\Theta\Theta i}\, dZ = \int_{-h/2}^{(h_c-h_s)/2} S_{(0)\Theta\Theta c}\, dZ + \int_{(h_c-h_s)/2}^{h/2} S_{(0)\Theta\Theta s}\, dZ, \tag{8.127}$$

as well as out-of-plane shear stress resultants defined by

$$Q_R \equiv \int_{-h/2}^{h/2} S_{(0)RZi}\, dZ = \int_{-h/2}^{(h_c-h_s)/2} S_{(0)RZc}\, dZ + \int_{(h_c-h_s)/2}^{h/2} S_{(0)RZs}\, dZ, \tag{8.128}$$

$$Q_\Theta \equiv \int_{-h/2}^{h/2} S_{(0)\Theta Zi}\, dZ = \int_{-h/2}^{(h_c-h_s)/2} S_{(0)\Theta Zc}\, dZ + \int_{(h_c-h_s)/2}^{h/2} S_{(0)\Theta Zs}\, dZ, \tag{8.129}$$

equations (8.122)–(8.124) reduce by this integration process (after multiplying thru by R) to

$$(RN_R)_{,R} - N_\Theta + N_{R\Theta,\Theta} = 0, \tag{8.130}$$

$$\left(R^2 N_{R\Theta}\right)_{,R} + R N_{\Theta,\Theta} = 0, \tag{8.131}$$

and

$$pR + \left[R\left(\omega_{,R} N_R + \frac{w_{,\Theta}}{R} N_{R\Theta} + Q_R\right)\right]_{,R} + \left(\omega_{,R} N_{R\Theta} + \frac{w_{,\Theta}}{R} N_\Theta + Q_\Theta\right)_{,\Theta} + \gamma_0 g R = 0, \tag{8.132}$$

where we have also applied the leading order boundary conditions of pressure (8.52). In the last equation we have introduced the pressure difference p, defined earlier in equation (8.75), and have assumed the mass densities to be constant through their respective thicknesses (although they, as well as p, may vary with R and Θ).

We next multiply equations (8.122) and (8.123) by Z, to obtain

$$\begin{aligned} &Z\left(S_{(0)RZi}\right)_{,Z} + \frac{1}{R}\left[\left(R\, Z\, S_{(0)RRi}\right)_{,R} - Z\, S_{(0)\Theta\Theta i} + \left(Z\, S_{(0)R\Theta i}\right)_{,\Theta}\right] \\ &\quad \equiv \left(Z\, S_{(0)RZi}\right)_{,Z} - S_{(0)RZi} + \frac{1}{R}\left[\left(R\, Z\, S_{(0)RRi}\right)_{,R} - Z\, S_{(0)\Theta\Theta i} + \left(Z\, S_{(0)R\Theta i}\right)_{,\Theta}\right] = 0, \end{aligned} \tag{8.133}$$

and

$$\begin{aligned} &Z\left(S_{(0)\Theta Zi}\right)_{,Z} + \frac{1}{R^2}\left(R^2\, Z\, S_{(0)R\Theta i}\right)_{,R} + \frac{1}{R}\left(Z\, S_{(0)\Theta\Theta i}\right)_{,\Theta} \\ &\quad \equiv \left(Z\, S_{(0)\Theta Zi}\right)_{,Z} - S_{(0)\Theta Zi} + \frac{1}{R^2}\left(R^2\, Z\, S_{(0)R\Theta i}\right)_{,R} + \frac{1}{R}\left(Z\, S_{(0)\Theta\Theta i}\right)_{,\Theta} = 0. \end{aligned} \tag{8.134}$$

Integrating each of these equations through the thickness, applying the boundary conditions of pressure again, and introducing the following stress couples:

$$M_R \equiv \int_{-h/2}^{h/2} Z\, S_{(0)RRi}\, dZ = \int_{-h/2}^{(h_c-h_s)/2} Z\, S_{(0)RRc}\, dZ + \int_{(h_c-h_s)/2}^{h/2} Z\, S_{(0)RRs}\, dZ, \tag{8.135}$$

$$M_{R\Theta} \equiv \int_{-h/2}^{h/2} Z\, S_{(0)R\Theta i}\, dZ = \int_{-h/2}^{(h_c-h_s)/2} Z\, S_{(0)R\Theta c}\, dZ + \int_{(h_c-h_s)/2}^{h/2} Z\, S_{(0)R\Theta s}\, dZ, \tag{8.136}$$

$$M_\Theta \equiv \int_{-h/2}^{h/2} Z\, S_{(0)\Theta\Theta i}\, dZ = \int_{-h/2}^{(h_c-h_s)/2} Z\, S_{(0)\Theta\Theta c}\, dZ + \int_{(h_c-h_s)/2}^{h/2} Z\, S_{(0)\Theta\Theta s}\, dZ, \tag{8.137}$$

we obtain from (8.133) and (8.134) the following equations involving the shear stress resultants and couples:

$$-Q_R + \frac{1}{R}\left[(R\, M_R)_{,R} - M_\Theta + M_{R\Theta,\Theta}\right] = 0, \tag{8.138}$$

and

$$-Q_\Theta + \frac{1}{R^2}\left(R^2\, M_{R\Theta}\right)_{,R} + \frac{1}{R} M_{\Theta,\Theta} = 0. \tag{8.139}$$

The last two equations can be used to eliminate Q_R and Q_Θ in equation (8.132), to bring it to the form

$$\begin{aligned} & \left[R\left(\omega_{,R}\, N_R + \frac{w_{,\Theta}}{R} N_{R\Theta}\right) + (R\, M_R)_{,R} - M_\Theta + M_{R\Theta,\Theta}\right]_{,R} \\ & \quad + \left[\omega_{,R} N_{R\Theta} + \frac{w_{,\Theta}}{R} N_\Theta + \frac{1}{R^2}\left(R^2\, M_{R\Theta}\right)_{,R} + \frac{1}{R}\, M_{\Theta,\Theta}\right]_{,\Theta} + (p + \gamma_0\, g)\, R = 0. \end{aligned} \tag{8.140}$$

Equations (8.130), (8.131), and (8.140) are the fundamental equilibrium equations in terms of stress resultants and couples, which are given by the integrals (8.125)–(8.127) and (8.135)–(8.137). Performing these integrals, we obtain (see Appendix A for the details):

$$N_R = \mathcal{N} + A\,\epsilon^0_{RR} + A_\nu\,\epsilon^0_{\Theta\Theta} + B\,k_{RR} + B_\nu\,k_{\Theta\Theta}, \tag{8.141}$$

$$N_{R\Theta} = A_\Theta\,\epsilon^0_{R\Theta} + B_\Theta\,k_{R\Theta}, \tag{8.142}$$

$$N_\Theta = \mathcal{N} + A_\nu\,\epsilon^0_{RR} + A\,\epsilon^0_{\Theta\Theta} + B_\nu\,k_{RR} + B\,k_{\Theta\Theta}, \tag{8.143}$$

$$M_R = -\mathcal{M} - B\,\epsilon^0_{RR} - B_\nu\,\epsilon^0_{\Theta\Theta} - D\,k_{RR} - D_\nu\,k_{\Theta\Theta}, \tag{8.144}$$

$$M_{R\Theta} = -B_\Theta\,\epsilon^0_{R\Theta} - D_\Theta\,k_{R\Theta}, \tag{8.145}$$

$$M_\Theta = -\mathcal{M} - B_\nu\,\epsilon^0_{RR} - B\,\epsilon^0_{\Theta\Theta} - D_\nu\,k_{RR} - D\,k_{\Theta\Theta}, \tag{8.146}$$

where we have introduced the following new constant (having units of $N \cdot m/m$):

$$\mathcal{M} = \frac{1}{2}\,h_c\,h_s\,(S_c - S_s), \tag{8.147}$$

all other constants having been defined in (8.116)–(8.121). Substitution of these expressions in equations (8.130), (8.131), and (8.140) yields equations (8.113)–(8.115), which were obtained in the previous Subsection as continuity conditions.

9 Geometrically Nonlinear Membrane Shell Laminate Theory

We follow Erbay [11], and set the scaling exponents of Section 7, Table 1, to $r = m = 1/2$, $\ell = n = 1$, $t = 3/2$, $p = 2$, $q = 5/2$, and $\mu = 1/2$, to obtain a generalization of the geometrically nonlinear membrane theory of Hencky [13] and Campbell [14] to a membrane laminate. With this choice of exponents, the constitutive equation (7.23) takes the form

$$\begin{aligned} \varepsilon^{5/2}\Sigma_i \widehat{S}_{iZZ} = \varepsilon \Sigma_i \widehat{S}^{nm}_i + \mathcal{E}_i \Bigg\{ (1-\nu_i) \left[\varepsilon^{-1/2}\,\widehat{W}_{,\zeta} + \frac{1}{2}\varepsilon^{-1}\,\widehat{W}^2_{,\zeta} + \left(\frac{\widehat{U}^2_{,\zeta} + \widehat{V}^2_{,\zeta}}{2}\right)\right] \\ + \nu_i \left[\varepsilon\left(\widehat{U}_{,\rho} + \frac{1}{2}\widehat{W}^2_{,\rho} + \frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho} + \frac{\widehat{W}^2_{,\Theta}}{2\rho^2}\right) + \varepsilon^2\left(\frac{\widehat{U}^2_{,\rho} + \widehat{V}^2_{,\rho}}{2} + \frac{(\widehat{U}_{,\Theta} - v)^2 + (\widehat{V}_{,\Theta} + \widehat{U})^2}{2\rho^2}\right)\right]\Bigg\}. \end{aligned} \tag{9.1}$$

Each variable is now expanded in an asymptotic series in powers of $\varepsilon^{1/2}$ (recall that $\mu = 1/2$ here), and in particular we have for the partial derivatives of $\widehat{W}$ with respect to ζ:

$$\widehat{W}_{,\zeta} = \widehat{W}_{(0),\zeta} + \varepsilon^{1/2}\,\widehat{W}_{(1),\zeta} + \varepsilon\,\widehat{W}_{(2),\zeta} + \varepsilon^{3/2}\,\widehat{W}_{(3),\zeta} + \varepsilon^2\,\widehat{W}_{(4)} + O(\varepsilon^{5/2}), \tag{9.2}$$

hence, to order ε^2, we find

$$\widehat{W}^2_{,\zeta} = \widehat{W}^2_{(0),\zeta} + 2\,\widehat{W}_{(0),\zeta}\,\widehat{W}_{(1),\zeta}\,\varepsilon^{1/2} + \left[\widehat{W}^2_{(1),\zeta} + 2\,\widehat{W}_{(0),\zeta}\,\widehat{W}_{(2),\zeta}\right]\varepsilon$$
$$+ 2\left[\widehat{W}_{(0),\zeta}\widehat{W}_{(3),\zeta} + \widehat{W}_{(1),\zeta}\,\widehat{W}_{(2),\zeta}\right]\varepsilon^{3/2} + \left[\widehat{W}^2_{(2),\zeta} + 2\widehat{W}_{(0),\zeta}\,\widehat{W}_{(4),\zeta} + 2\widehat{W}_{(1),\zeta}\,\widehat{W}_{(3),\zeta}\right]\varepsilon^2 + O(\varepsilon^{5/2}), \tag{9.3}$$

and similar expressions for $\widehat{U}^2_{,\zeta}$ and $\widehat{V}^2_{,\zeta}$. Substituting in (9.1), we obtain to order ε:

$$\begin{aligned}\varepsilon^{5/2}\Sigma_i\widehat{S}_{(0)iZZ} = {} & \varepsilon\Sigma_i\widehat{S}^{nm}_i + \mathcal{E}_i\Bigg\{(1-\nu_i)\Bigg[\frac{1}{2}\varepsilon^{-1}\,\widehat{W}^2_{(0),\zeta} + \varepsilon^{-1/2}\left(\widehat{W}_{(0),\zeta} + \widehat{W}_{(0),\zeta}\widehat{W}_{(1),\zeta}\right)\\
& + \frac{1}{2}\left(2\,\widehat{W}_{(1),\zeta} + \widehat{W}^2_{(1),\zeta} + 2\,\widehat{W}_{(0),\zeta}\,\widehat{W}_{(2),\zeta} + \widehat{U}^2_{(0),\zeta} + \widehat{V}^2_{(0),\zeta}\right)\\
& + \varepsilon^{1/2}\left(\widehat{W}_{(2),\zeta} + \widehat{W}_{(0),\zeta}\,\widehat{W}_{(3),\zeta} + \widehat{W}_{(1),\zeta}\,\widehat{W}_{(2),\zeta} + \widehat{U}_{(0),\zeta}\widehat{U}_{(1),\zeta} + \widehat{V}_{(0),\zeta}\widehat{V}_{(1),\zeta}\right)\\
& + \frac{1}{2}\varepsilon\left(2\,\widehat{W}_{(3),\zeta} + \widehat{W}^2_{(2),\zeta} + 2\widehat{W}_{(1),\zeta}\widehat{W}_{(3),\zeta} + \widehat{U}^2_{(1),\zeta} + 2\widehat{U}_{(0),\zeta}\widehat{U}_{(2),\zeta} + \widehat{V}^2_{(1),\zeta} + 2\widehat{V}_{(0),\zeta}\widehat{V}_{(2),\zeta}\right)\Bigg]\\
& + \nu_i\,\varepsilon\left(\widehat{U}_{(0),\rho} + \frac{1}{2}\widehat{W}^2_{(0),\rho} + \frac{\widehat{V}_{(0),\Theta} + \widehat{U}_{(0)}}{\rho} + \frac{\widehat{W}^2_{(0),\Theta}}{2\rho^2}\right)\Bigg\} + O(\varepsilon^{3/2}).\end{aligned} \tag{9.4}$$

Since this expression must hold in the limit $\varepsilon \to 0$ (vanishing thickness), the coefficients of negative powers of ε must vanish, hence $\widehat{W}_{(0),\zeta} = 0$, yielding the solution

$$\widehat{W}_{(0)} = w(\rho, \Theta), \tag{9.5}$$

where w is an arbitrary function of ρ and Θ only. Equation (9.4) then reduces to

$$\begin{aligned}\varepsilon^{5/2}\Sigma_i\widehat{S}_{(0)iZZ} = {} & \varepsilon\Sigma_i\widehat{S}^{nm}_i + \mathcal{E}_i\Bigg\{(1-\nu_i)\Bigg[\frac{1}{2}\left(2\,\widehat{W}_{(1),\zeta} + \widehat{W}^2_{(1),\zeta} + \widehat{U}^2_{(0),\zeta} + \widehat{V}^2_{(0),\zeta}\right)\\
& + \varepsilon^{1/2}\left(\widehat{W}_{(2),\zeta} + \widehat{W}_{(1),\zeta}\,\widehat{W}_{(2),\zeta} + \widehat{U}_{(0),\zeta}\widehat{U}_{(1),\zeta} + \widehat{V}_{(0),\zeta}\widehat{V}_{(1),\zeta}\right)\\
& + \frac{1}{2}\varepsilon\left(2\,\widehat{W}_{(3),\zeta} + \widehat{W}^2_{(2),\zeta} + 2\widehat{W}_{(1),\zeta}\widehat{W}_{(3),\zeta} + \widehat{U}^2_{(1),\zeta} + 2\widehat{U}_{(0),\zeta}\widehat{U}_{(2),\zeta} + \widehat{V}^2_{(1),\zeta} + 2\widehat{V}_{(0),\zeta}\widehat{V}_{(2),\zeta}\right)\Bigg]\\
& + \nu_i\,\varepsilon\left(\widehat{U}_{(0),\rho} + \frac{1}{2}\widehat{W}^2_{(0),\rho} + \frac{\widehat{V}_{(0),\Theta} + \widehat{U}_{(0)}}{\rho} + \frac{\widehat{W}^2_{(0),\Theta}}{2\rho^2}\right)\Bigg\} + O(\varepsilon^{3/2}).\end{aligned} \tag{9.6}$$

The following three additional conditions are obtained from the vanishing of the coefficients of ε^0, $\varepsilon^{1/2}$, and ε, respectively:

$$2\widehat{W}_{(1),\zeta} + \widehat{W}^2_{(1),\zeta} + \widehat{U}^2_{(0),\zeta} + \widehat{V}^2_{(0),\zeta} = 0, \tag{9.7}$$

$$\widehat{W}_{(2),\zeta} + \widehat{W}_{(1),\zeta}\,\widehat{W}_{(2),\zeta} + \widehat{U}_{(0),\zeta}\widehat{U}_{(1),\zeta} + \widehat{V}_{(0),\zeta}\widehat{V}_{(1),\zeta} = 0, \tag{9.8}$$

$$\begin{aligned}2\,\Sigma_i\widehat{S}^{nm}_i + \mathcal{E}_i\Bigg[&(1-\nu_i)\left(2\,\widehat{W}_{(3),\zeta} + \widehat{W}^2_{(2),\zeta} + 2\,\widehat{W}_{(1),\zeta}\widehat{W}_{(3),\zeta} + \widehat{U}^2_{(1),\zeta} + 2\,\widehat{U}_{(0),\zeta}\widehat{U}_{(2),\zeta} + \widehat{V}^2_{(1),\zeta} + 2\widehat{V}_{(0),\zeta}\widehat{V}_{(2),\zeta}\right)\\
& + 2\,\nu_i\left(\widehat{U}_{(0),\rho} + \frac{1}{2}w^2_{,\rho} + \frac{\widehat{V}_{(0),\Theta} + \widehat{U}_{(0)}}{\rho} + \frac{w^2_{,\Theta}}{2\rho^2}\right)\Bigg] = 0,\end{aligned} \tag{9.9}$$

which we shall need later. We turn next to the two off-diagonal constitutive relations (7.24) and (7.25) under these new scalings:

$$\varepsilon^2\Sigma_i\widehat{S}_{RZi} = \frac{G_i}{2}\left[\widehat{U}_{,\zeta} + \varepsilon^{1/2}\widehat{W}_{,\rho} + \widehat{\Gamma}_{,\rho}\widehat{W}_{,\zeta} + \widehat{W}_{,\rho}\widehat{W}_{,\zeta} + \varepsilon\,(\widehat{U}_{,\rho}\widehat{U}_{,\zeta} + \widehat{V}_{,\rho}\widehat{V}_{,\zeta})\right], \tag{9.10}$$

$$\varepsilon^2\Sigma_i\widehat{S}_{\Theta Zi} = \frac{G_i}{2}\left\{\widehat{V}_{,\zeta} + \varepsilon^{1/2}\frac{\widehat{W}_{,\Theta}}{\rho} + \frac{\widehat{W}_{,\zeta}\widehat{W}_{,\Theta}}{\rho} + \varepsilon\left[\frac{\widehat{U}_{,\zeta}(\widehat{U}_{,\Theta} - \widehat{V}) + \widehat{V}_{,\zeta}(\widehat{V}_{,\Theta} + \widehat{U})}{\rho}\right]\right\}. \tag{9.11}$$

The leading order terms in these equations imply $\widehat{U}_{(0),\zeta} = 0$ and $\widehat{V}_{(0),\zeta} = 0$, hence the leading order in-plane displacement components are both independent of ζ, i.e.,

$$\widehat{U}_{(0)} = u(\rho,\Theta), \qquad \widehat{V}_{(0)} = v(\rho,\Theta). \tag{9.12}$$

The conditions $\widehat{U}_{(0),\zeta} = 0$ and $\widehat{V}_{(0),\zeta} = 0$ reduce equations (9.7)–(9.9) to

$$\widehat{W}_{(1),\zeta}\left(2 + \widehat{W}_{(1),\zeta}\right) = 0, \tag{9.13}$$

$$\widehat{W}_{(2),\zeta}\left(1 + \widehat{W}_{(1),\zeta}\right) = 0, \tag{9.14}$$

$$\begin{aligned} 2\,\Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\Bigg[&(1-\nu_i)\left(2\,\widehat{W}_{(3),\zeta} + \widehat{W}^2_{(2),\zeta} + 2\,\widehat{W}_{(1),\zeta}\widehat{W}_{(3),\zeta} + \widehat{U}^2_{(1),\zeta} + \widehat{V}^2_{(1),\zeta}\right) \\ &+ 2\,\nu_i\left(u_{,\rho} + \frac{1}{2}w^2_{,\rho} + \frac{v_{,\Theta} + u}{\rho} + \frac{w^2_{,\Theta}}{2\rho^2}\right)\Bigg] = 0. \end{aligned} \tag{9.15}$$

Noting that $\widehat{W}_{(0),\zeta} = 0$ implies a scaled Jacobian determinant (7.22) of the form

$$J = 1 + \widehat{W}_{(1),\zeta} + O(\varepsilon^{1/2}),$$

it follows that in the limit $\varepsilon \to 0$, since we must have $J > 0$ in that limit, we must also require $\widehat{W}_{1,\zeta} > -1$. This precludes the solution $\widehat{W}_{(1),\zeta} = -2$ of (9.13), implying that $\widehat{W}_{(1),\zeta} = 0$. This leads to $\widehat{W}_{(2),\zeta} = 0$, from (9.14), hence (9.15) yields the following expression for $\widehat{W}_{(3),\zeta}$:

$$\widehat{W}_{(3),\zeta} = -\frac{1}{2}\left(\widehat{U}^2_{(1),\zeta} + \widehat{V}^2_{(1),\zeta}\right) - \frac{\Sigma_i\widehat{S}_i^{nm}}{\mathcal{E}_i\,(1-\nu_i)} - \frac{\nu_i}{1-\nu_i}\left(u_{,\rho} + \frac{1}{2}w^2_{,\rho} + \frac{v_{,\Theta} + u}{\rho} + \frac{w^2_{,\Theta}}{2\rho^2}\right). \tag{9.16}$$

Next, the off-diagonal in-plane constitutive relation (7.26) has the following form under these scalings:

$$\begin{aligned} \varepsilon\Sigma_i\widehat{S}_{R\Theta i} = \frac{G_i}{2}\Bigg\{&\varepsilon\left(\widehat{V}_{,\rho} + \frac{\widehat{U}_{,\Theta} - \widehat{V}}{\rho}\right) + \varepsilon\,\frac{\widehat{\Gamma}_{,\rho}\widehat{W}_{,\Theta}}{\rho} + \varepsilon\frac{\widehat{W}_{,\rho}\widehat{W}_{,\Theta}}{\rho} \\ &+ \varepsilon^2\left[\frac{\widehat{U}_{,\rho}(\widehat{U}_{,\Theta} - \widehat{V}) + \widehat{V}_{,\rho}(\widehat{V}_{,\Theta} + \widehat{U})}{\rho}\right]\Bigg\}. \end{aligned} \tag{9.17}$$

The leading order term here yields the same constitutive relation found in (8.18), except that the strain-displacement relation has changed:

$$\Sigma_i\widehat{S}_{(0)R\Theta i} = \frac{G_i}{2}\left(v_{,\rho} + \frac{u_{,\Theta} - v}{\rho} + \frac{\widehat{\Gamma}_{,\rho}w_{,\Theta}}{\rho} + \frac{w_{,\rho}w_{,\Theta}}{\rho}\right) \equiv G_i\,\widehat{\epsilon}_{R\Theta}, \tag{9.18}$$

where

$$\widehat{\epsilon}_{R\Theta} \equiv \frac{1}{2}\left(v_{,\rho} + \frac{u_{,\Theta} - v}{\rho} + \frac{\widehat{\Gamma}_{,\rho} w_{,\Theta}}{\rho} + \frac{w_{,\rho} w_{,\Theta}}{\rho}\right). \tag{9.19}$$

The final two constitutive relations (7.27) and (7.28) take the following forms:

$$\begin{aligned}
\varepsilon\Sigma_i\widehat{S}_{RRi} &= \varepsilon\Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\Bigg((1-\nu_i)\left\{\varepsilon\widehat{U}_{,\rho} + \varepsilon\widehat{\Gamma}_{,\rho}\widehat{W}_{,\rho} + \frac{1}{2}\left[\varepsilon\widehat{W}_{,\rho}^2 + \varepsilon^2\left(\widehat{U}_{,\rho}^2 + \widehat{V}_{,\rho}^2\right)\right]\right\} \\
&+ \nu_i\left\{\varepsilon\left(\frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho}\right) + \varepsilon\frac{\widehat{W}_{,\Theta}^2}{2\rho^2} + \varepsilon^2\left[\frac{(\widehat{U}_{,\Theta} - \widehat{V})^2 + (\widehat{V}_{,\Theta} + \widehat{U})^2}{2\rho^2}\right]\right. \\
&\left.+ \varepsilon^{-1/2}\widehat{W}_{,\zeta} + \varepsilon^{-1}\frac{1}{2}\widehat{W}_{,\zeta}^2 + \frac{1}{2}\left(\widehat{U}_{,\zeta}^2 + \widehat{V}_{,\zeta}^2\right)\right\}\Bigg),
\end{aligned} \tag{9.20}$$

and

$$\begin{aligned}
\varepsilon\Sigma_i\widehat{S}_{\Theta\Theta i} &= \varepsilon\Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\Bigg((1-\nu_i)\left\{\varepsilon\left(\frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho}\right) + \varepsilon\frac{\widehat{W}_{,\Theta}^2}{2\rho^2} + \varepsilon^2\left[\frac{(\widehat{U}_{,\Theta} - \widehat{V})^2 + (\widehat{V}_{,\Theta} + \widehat{U})^2}{2\rho^2}\right]\right\} \\
&+ \nu_i\left\{\varepsilon\widehat{U}_{,\rho} + \varepsilon\widehat{\Gamma}_{,\rho}\widehat{W}_{,\rho} + \frac{1}{2}\left[\varepsilon\widehat{W}_{,\rho}^2 + \varepsilon^2\left(\widehat{U}_{,\rho}^2 + \widehat{V}_{,\rho}^2\right)\right]\right. \\
&\left.+ \varepsilon^{-1/2}\widehat{W}_{,\zeta} + \varepsilon^{-1}\frac{1}{2}\widehat{W}_{,\zeta}^2 + \frac{1}{2}\left(\widehat{U}_{,\zeta}^2 + \widehat{V}_{,\zeta}^2\right)\right\}\Bigg).
\end{aligned} \tag{9.21}$$

In the last two equations the terms up to $O(\varepsilon)$ involving derivatives with respect to ζ reduce to simply $\widehat{W}_{(3),\zeta}$, which can be replaced by (9.16). The coefficients of ε then yield, after some algebra, the same in-plane constitutive relations (8.27) and (8.28), except that the strains are now given by the Z-independent expressions

$$\widehat{\epsilon}_{RR} \equiv u_{,\rho} + \frac{1}{2}w_{,\rho}^2, \qquad \widehat{\epsilon}_{\Theta\Theta} \equiv \frac{v_{,\Theta} + u}{\rho} + \frac{w_{,\Theta}^2}{2\rho^2}. \tag{9.22}$$

The most important results here are that the leading order in-plane displacement components have the forms $U_{(0)R} = u(R,\Theta)$ and $U_{(0)\Theta} = v(R,\Theta)$ (in terms of the physical variables), i.e., they are independent of Z. This is in marked contrast to the Kirchhoff-Love expressions (8.31) found previously, which are linear in Z. As a result of these simple expressions for the in-plane displacements, the curvature terms k_{RR}, $k_{R\Theta}$, and $k_{\Theta\Theta}$ do not appear in equations (8.35)–(8.37) defining the in-plane strain components. Summarizing to this point, we have derived the following expressions from the scaled constitutive relations:

$$U_{(0)Z} = w(R,\Theta), \qquad U_{(0)R} = u(R,\Theta), \qquad U_{(0)\Theta} = v(R,\Theta), \tag{9.23}$$

$$S_{(0)R\Theta i} = G_i\,\epsilon_{R\Theta}, \tag{9.24}$$

$$S_{(0)RRi} = S_i + \frac{E_i}{1-\nu_i^2}\left(\epsilon_{RR} + \nu_i\epsilon_{\Theta\Theta}\right), \tag{9.25}$$

$$S_{(0)\Theta\Theta i} = S_i + \frac{E_i}{1-\nu_i^2}\left(\epsilon_{\Theta\Theta} + \nu_i\epsilon_{RR}\right), \tag{9.26}$$

where $\widehat{S}_i$ is given by equation (8.29), and

$$\epsilon_{R\Theta} = \frac{1}{2}\left(v_{,R} + \frac{u_{,\Theta} - v}{R} + \frac{\Gamma_{,R} w_{,\Theta}}{R} + \frac{w_{,R} w_{,\Theta}}{R}\right) \equiv \epsilon_{R\Theta}^0, \tag{9.27}$$

$$\epsilon_{RR} = u_{,R} + \Gamma_{,R} w_{,R} + \frac{1}{2} w_{,R}^2 \equiv \epsilon_{RR}^0, \tag{9.28}$$

$$\epsilon_{\Theta\Theta} = \frac{v_{,\Theta} + u}{R} + \frac{w_{,\Theta}^2}{2R^2} \equiv \epsilon_{\Theta\Theta}^0. \tag{9.29}$$

Under these new scalings, the radial and circumferential equilibrium equations (8.122) and (8.123) are unchanged, but the shear stresses disappear from the terms of the axial equilibrium equation (8.124) involving derivatives with respect to R and Θ. The leading order equilibrium equations are then given by

$$S_{(0)RZi,Z} + \frac{1}{R}\left[\left(RS_{(0)RRi}\right)_{,R} - S_{(0)\Theta\Theta i} + S_{(0)R\Theta i,\Theta}\right] = 0, \tag{9.30}$$

$$S_{(0)\Theta Zi,Z} + \frac{1}{R^2}\left(R^2 S_{(0)R\Theta i}\right)_{,R} + \frac{1}{R} S_{(0)\Theta\Theta i,\Theta} = 0. \tag{9.31}$$

$$\begin{aligned}\left(\omega_{,R} S_{(0)RZi} + \frac{w_{,\Theta}}{R} S_{(0)\Theta Zi} + S_{(0)ZZi}\right)_{,Z} + \frac{1}{R}\left(\omega_{,R} S_{(0)R\Theta i} + \frac{w_{,\Theta}}{R} S_{(0)\Theta\Theta i}\right)_{,\Theta} \\ + \frac{1}{R}\left[R\left(\omega_{,R} S_{(0)RRi} + \frac{w_{,\Theta}}{R} S_{(0)R\Theta i}\right)\right]_{,R} + \rho_{0i}\, g = 0,\end{aligned} \tag{9.32}$$

where we used the result $\widehat{W}_{,\varsigma} = \varepsilon^{3/2}\widehat{W}_{(3),\varsigma} + O(\varepsilon^2)$ in obtaining the scaled version of the last equation. The boundary conditions of pressure are again given by (8.53), so the equilibrium equations can be integrated through the thickness, applying the boundary conditions of pressure, to obtain (after multiplying thru by R) equations in terms of the in-plane stress resultants (8.125)–(8.127):

$$\left(RN_R\right)_{,R} - N_\Theta + N_{R\Theta,\Theta} = 0, \tag{9.33}$$

$$\left(R^2 N_{R\Theta}\right)_{,R} + R N_{\Theta,\Theta} = 0, \tag{9.34}$$

$$\left[R\left(\omega_{,R} N_R + \frac{w_{,\Theta}}{R} N_{R\Theta}\right)\right]_{,R} + \left(\omega_{,R} N_{R\Theta} + \frac{w_{,\Theta}}{R} N_\Theta\right)_{,\Theta} + \left(p + \gamma_0\, g\right) R = 0. \tag{9.35}$$

Since the curvatures have vanished from the formulation, the stress resultants (8.141)–(8.143) in this case reduce to

$$N_R = \mathcal{N} + A\,\epsilon_{RR}^0 + A_\nu\,\epsilon_{\Theta\Theta}^0, \tag{9.36}$$

$$N_{R\Theta} = A_\Theta\,\epsilon_{R\Theta}^0, \tag{9.37}$$

$$N_\Theta = \mathcal{N} + A_\nu\,\epsilon_{RR}^0 + A\,\epsilon_{\Theta\Theta}^0. \tag{9.38}$$

10 Geometrically Linear Shell Laminate Theory

Assuming the scaling exponents $r = 1$, $m = 2$, $\ell = n = 3$, $t = p = 4$, $q = 5$, and $\mu = 1$, we begin again with the constitutive relation (5.5) for S_{ZZ}, from which we obtain

$$\begin{aligned}\varepsilon^5 \Sigma_i \widehat{S}_{ZZi} &= \varepsilon^3 \Sigma_i \widehat{S}_i^{nm} + \mathcal{E}_i\Big\{ (1-\nu_i)\left[\varepsilon \widehat{W}_{,\varsigma} + \varepsilon^2 \frac{1}{2}\widehat{W}_{,\varsigma}^2 + \varepsilon^4 \frac{1}{2}\left(\widehat{U}_{,\varsigma}^2 + \widehat{V}_{,\varsigma}^2\right)\right] \\ &+ \nu_i \Big[\varepsilon^3\left(\widehat{U}_{,\rho} + \widehat{\Gamma}_{,\rho}\widehat{W}_{,\rho} + \frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho}\right) + \varepsilon^4\left(\frac{1}{2}\widehat{W}_{,\rho}^2 + \frac{\widehat{W}_{,\Theta}^2}{2\rho^2}\right) \\ &+ \varepsilon^6\left(\frac{\widehat{U}_{,\rho}^2 + \widehat{V}_{,\rho}^2}{2} + \frac{(\widehat{U}_{,\Theta} - \widehat{V})^2 + (\widehat{V}_{,\Theta} + \widehat{U})^2}{2\rho^2}\right)\Big]\Big\}.\end{aligned} \tag{10.1}$$

Each of the scaled stress tensor components, as well as each of the displacement components and its partial derivatives, is now written as an asymptotic expansion in $\delta = \varepsilon$, since $\mu = 1$. Substituting the appropriate expansions for the variables appearing in equation (10.1), we obtain to third order on the right-hand side:

$$\varepsilon^5 \Sigma_i \widehat{S}_{(0)ZZi} = \varepsilon^3 \Sigma_i \widehat{S}_i^{nm} + \mathcal{E}_i \Big\{ (1-\nu_i) \left[\varepsilon \widehat{W}_{(0),\zeta} + \varepsilon^2 \left(\frac{1}{2} \widehat{W}^2_{(0),\zeta} + \widehat{W}_{(1),\zeta} \right) \right]$$

$$+ \ \varepsilon^3 \left[(1-\nu_i) \left(\widehat{W}_{(2),\zeta} + \widehat{W}_{(0),\zeta} \widehat{W}_{(1),\zeta} \right) + \nu_i \left(\widehat{U}_{(0),\rho} + \widehat{\Gamma}_{,\rho} \widehat{W}_{(0),\rho} + \frac{\widehat{V}_{(0),\Theta} + \widehat{U}_{(0)}}{\rho} \right) \right] + O(\varepsilon^4) \Big\} . \quad (10.2)$$

The leading order term of this relation yields

$$\widehat{W}_{(0),\zeta} = 0 \quad \Rightarrow \quad \widehat{W}_{(0)} = \widehat{w}(\rho, \Theta), \quad (10.3)$$

where $\widehat{w}$ is an arbitrary function of ρ and Θ only, and then the next order term reduces to the result $\widehat{W}_{(1),\zeta} = 0$. With these results, the third order term then gives the following important relation:

$$\widehat{W}_{(2),\zeta} = -\frac{\Sigma_i}{\mathcal{E}_i(1-\nu_i)} \widehat{S}_i^{nm} - \frac{\nu_i}{1-\nu_i} \left(\widehat{U}_{(0),\rho} + \widehat{\Gamma}_{,\rho} \widehat{w}_{,\rho} + \frac{\widehat{V}_{(0),\Theta} + \widehat{U}_{(0)}}{\rho} \right). \quad (10.4)$$

Next, consider the last two off-diagonal constitutive relations of (5.6) for the out-of-plane stress components. The scaling of these components yields the scaled relations (to third order on the right-hand sides):

$$\varepsilon^4 \Sigma_i \widehat{S}_{RZi} = \frac{G_i}{2} \left[\varepsilon^2 \left(\widehat{U}_{(0),\zeta} + \widehat{W}_{(0),\rho} + \widehat{\Gamma}_{,\rho} \widehat{W}_{(0),\zeta} \right) + O(\varepsilon^3) \right], \quad (10.5)$$

$$\varepsilon^4 \Sigma_i \widehat{S}_{\Theta Zi} = \frac{G_i}{2} \left[\varepsilon^2 \left(\widehat{V}_{(0),\zeta} + \frac{\widehat{W}_{(0),\Theta}}{\rho} \right) + O(\varepsilon^3) \right], \quad (10.6)$$

respectively. From the leading order terms of these expressions, recalling that $\widehat{W}_{(0),\zeta} = 0$, we obtain the following two equations:

$$\widehat{U}_{(0),\zeta} + \widehat{w}_{,\rho} = 0, \qquad \widehat{V}_{(0),\zeta} + \frac{\widehat{w}_{,\Theta}}{\rho} = 0,$$

which may be integrated to obtain

$$\widehat{U}_{(0)} = \widehat{u}(\rho, \Theta) - \zeta \, \widehat{w}_{,\rho}, \qquad \text{and} \qquad \widehat{V}_{(0)} = \widehat{v}(\rho, \Theta) - \zeta \frac{\widehat{w}_{,\Theta}}{\rho}, \quad (10.7)$$

where $\widehat{u}$ and $\widehat{v}$ are arbitrary functions of ρ and Θ only.

The scaled version of the first off-diagonal in-plane constitutive relation of (5.6) reduces to

$$\varepsilon^3 \Sigma_i \widehat{S}_{R\Theta i} = G_i \left[\varepsilon^3 \frac{1}{2} \left(\widehat{V}_{(0),\rho} + \frac{\widehat{U}_{(0),\Theta} - \widehat{V}_{(0)}}{\rho} + \frac{\widehat{\Gamma}_{,\rho} \widehat{W}_{(0),\Theta}}{\rho} \right) + O(\varepsilon^4) \right].$$

Thus, to leading order we obtain the simple relation

$$\Sigma_i \widehat{S}_{(0)R\Theta i} = G_i \, \widehat{\epsilon}_{R\Theta}, \quad (10.8)$$

after introducing an in-plane strain component $\widehat{\epsilon}_{R\Theta}$ defined by

$$\widehat{\epsilon}_{R\Theta} \equiv \frac{1}{2} \left(\widehat{V}_{(0),\rho} + \frac{\widehat{U}_{(0),\Theta} - \widehat{V}_{(0)}}{\rho} + \frac{\widehat{\Gamma}_{,\rho} \widehat{w}_{,\Theta}}{\rho} \right)$$

$$= \frac{1}{2} \left[\widehat{v}_{,\rho} + \frac{\widehat{u}_{,\Theta} - \widehat{v}}{\rho} + \frac{\widehat{\Gamma}_{,\rho} \widehat{w}_{,\Theta}}{\rho} + 2\zeta \left(\frac{\widehat{w}_{,\Theta}}{\rho^2} - \frac{\widehat{w}_{,\rho\Theta}}{\rho} \right) \right]. \quad (10.9)$$

The final two constitutive relations (5.3) and (5.4) take the following forms (to third order in ε), after scaling of the various components:

$$
\begin{aligned}
\varepsilon^3\Sigma_i\widehat{S}_{RRi} &= \varepsilon^3\Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\Big\{(1-\nu_i)\,\varepsilon^3\left(\widehat{U}_{(0),\rho}+\widehat{\Gamma}_{,\rho}\widehat{W}_{(0),\rho}\right)\\
&+ \nu_i\Big[\varepsilon^3\left(\frac{\widehat{V}_{(0),\Theta}+\widehat{U}_{(0)}}{\rho}\right) + \varepsilon\left(\widehat{W}_{(0),\zeta}+\varepsilon\widehat{W}_{(1),\zeta}+\varepsilon^2\widehat{W}_{(2),\zeta}\right)\\
&+ \frac{1}{2}\varepsilon^2\left(\widehat{W}^2_{(0),\zeta}+2\varepsilon\widehat{W}_{(0),\zeta}\widehat{W}_{(1),\zeta}\right)\Big] + O(\varepsilon^4)\Big\},
\end{aligned}
\tag{10.10}
$$

and

$$
\begin{aligned}
\varepsilon^3\Sigma_i\widehat{S}_{\Theta\Theta i} &= \varepsilon^3\Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\Big\{(1-\nu_i)\,\varepsilon^3\left(\frac{\widehat{V}_{(0),\Theta}+\widehat{U}_{(0)}}{\rho}\right)\\
&+ \nu_i\Big[\varepsilon^3\left(\widehat{U}_{(0),\rho}+\widehat{\Gamma}_{,\rho}\widehat{W}_{(0),\rho}\right) + \varepsilon\left(\widehat{W}_{(0),\zeta}+\varepsilon\widehat{W}_{(1),\zeta}+\varepsilon^2\widehat{W}_{(2),\zeta}\right)\\
&+ \frac{1}{2}\varepsilon^2\left(\widehat{W}^2_{(0),\zeta}+2\varepsilon\widehat{W}_{(0),\zeta}\widehat{W}_{(1),\zeta}\right)\Big] + O(\varepsilon^4)\Big\}.
\end{aligned}
\tag{10.11}
$$

The first and second-order terms involving $\widehat{W}_{,\zeta}$ in these expressions vanish since $\widehat{W}_{(0),\zeta} = \widehat{W}_{(1),\zeta} = 0$, and (10.4) can be used to replace the third order term involving $\widehat{W}_{(2),\zeta}$. To simplify the reduction of the leading order expressions that follow, we introduce in-plane strain components $\widehat{\epsilon}_{RR}$ and $\widehat{\epsilon}_{\Theta\Theta}$ defined by

$$
\widehat{\epsilon}_{RR} \equiv \widehat{U}_{(0),\rho} + \widehat{\Gamma}_{,\rho}\widehat{w}_{,\rho} = \widehat{u}_{,\rho} + \widehat{\Gamma}_{,\rho}\widehat{w}_{,\rho} - \zeta\,\widehat{w}_{,\rho\rho} \equiv \widehat{\epsilon}^0_{RR} - \zeta\widehat{k}_{RR},
\tag{10.12}
$$

$$
\widehat{\epsilon}_{\Theta\Theta} \equiv \frac{\widehat{V}_{(0),\Theta} + \widehat{U}_{(0)}}{\rho} = \frac{\widehat{v}_{,\Theta}+\widehat{u}}{\rho} - \zeta\left(\frac{\widehat{w}_{,\rho}}{\rho} + \frac{\widehat{w}_{,\Theta\Theta}}{\rho^2}\right) \equiv \widehat{\epsilon}^0_{\Theta\Theta} - \zeta\widehat{k}_{\Theta\Theta},
\tag{10.13}
$$

where we have introduced ζ-independent terms

$$
\widehat{\epsilon}^0_{RR} \equiv \widehat{u}_{,\rho} + \widehat{\Gamma}_{,\rho}\widehat{w}_{,\rho}, \qquad \widehat{k}_{RR} \equiv \widehat{w}_{,\rho\rho},
\tag{10.14}
$$

$$
\widehat{\epsilon}^0_{\Theta\Theta} \equiv \frac{\widehat{v}_{,\Theta}+\widehat{u}}{\rho}, \qquad \widehat{k}_{\Theta\Theta} \equiv \frac{\widehat{w}_{,\rho}}{\rho} + \frac{\widehat{w}_{,\Theta\Theta}}{\rho^2}.
\tag{10.15}
$$

These, together with (10.4), allow us to write the leading order relations as

$$
\begin{aligned}
\Sigma_i\widehat{S}_{(0)RRi} &= \left(\frac{1-2\nu_i}{1-\nu_i}\right)\Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\left[(1-\nu_i)\widehat{\epsilon}_{RR} + \nu_i\widehat{\epsilon}_{\Theta\Theta} - \frac{\nu_i^2}{1-\nu_i}\left(\widehat{\epsilon}_{RR}+\widehat{\epsilon}_{\Theta\Theta}\right)\right]\\
&\equiv \left(\frac{1-2\nu_i}{1-\nu_i}\right)\Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\left(\frac{1-2\nu_i}{1-\nu_i}\right)\left(\widehat{\epsilon}_{RR}+\nu_i\widehat{\epsilon}_{\Theta\Theta}\right),\\
\Sigma_i\widehat{S}_{(0)\Theta\Theta i} &= \left(\frac{1-2\nu_i}{1-\nu_i}\right)\Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\left[(1-\nu_i)\widehat{\epsilon}_{\Theta\Theta} + \nu_i\widehat{\epsilon}_{RR} - \frac{\nu_i^2}{1-\nu_i}\left(\widehat{\epsilon}_{RR}+\widehat{\epsilon}_{\Theta\Theta}\right)\right]\\
&\equiv \left(\frac{1-2\nu_i}{1-\nu_i}\right)\Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\left(\frac{1-2\nu_i}{1-\nu_i}\right)\left(\widehat{\epsilon}_{\Theta\Theta}+\nu_i\widehat{\epsilon}_{RR}\right).
\end{aligned}
$$

Replacing $\mathcal{E}_i$ in the last two expressions yields

$$
\Sigma_i\widehat{S}_{(0)RRi} = \Sigma_i\,\widehat{S}_i + \frac{E_i}{1-\nu_i^2}\left(\widehat{\epsilon}_{RR} + \nu_i\widehat{\epsilon}_{\Theta\Theta}\right),
\tag{10.16}
$$

$$\Sigma_i \widehat{S}_{(0)\Theta\Theta i} = \Sigma_i \widehat{S}_i + \frac{E_i}{1-\nu_i^2} \left(\widehat{\epsilon}_{\Theta\Theta} + \nu_i \widehat{\epsilon}_{RR} \right), \tag{10.17}$$

where we have introduced the new constants

$$\widehat{S}_i \equiv \left(\frac{1-2\nu_i}{1-\nu_i} \right) \widehat{S}_i^{nm}. \tag{10.18}$$

We summarize to this point by rewriting the important results in terms of leading order variables that are functions of the physical coordinates R, Θ, and Z, viz.,

$$u \equiv \varepsilon^3 a\widehat{u}, \qquad v \equiv \varepsilon^3 a\widehat{v}, \qquad w \equiv \varepsilon^2 a\widehat{w}, \qquad \Gamma \equiv \varepsilon a\widehat{\Gamma}, \tag{10.19}$$

$$S_{(0)R\Theta i} \equiv \varepsilon^3 \Sigma_i \widehat{S}_{(0)R\Theta i} = G_i \, \epsilon_{R\Theta}. \tag{10.20}$$

$$S_{(0)RRi} \equiv \varepsilon^3 \Sigma_i \widehat{S}_{(0)RRi} = S_i + \frac{E_i}{1-\nu_i^2} \left(\epsilon_{RR} + \nu_i \epsilon_{\Theta\Theta} \right), \tag{10.21}$$

$$S_{(0)\Theta\Theta i} \equiv \varepsilon^3 \Sigma_i \widehat{S}_{(0)\Theta\Theta i} = S_i + \frac{E_i}{1-\nu_i^2} \left(\epsilon_{\Theta\Theta} + \nu_i \epsilon_{RR} \right), \tag{10.22}$$

where

$$S_i \equiv \varepsilon^3 \, \Sigma_i \widehat{S}_i = \left(\frac{1-2\nu_i}{1-\nu_i} \right) S_i^{nm} = -\left(\frac{E_i}{1-\nu_i} \right) \epsilon_i^{nm}, \tag{10.23}$$

and the last equality of (10.23) holds for the special case of thermal stress, following from (5.2). The strain components are given by

$$\epsilon_{R\Theta} \equiv \varepsilon^3 \widehat{\epsilon}_{R\Theta} = \frac{1}{2} \left[v_{,R} + \frac{u_{,\Theta} - v}{R} + \frac{\Gamma_{,R} w_{,\Theta}}{R} + 2Z \left(\frac{w_{,\Theta}}{R^2} - \frac{w_{,R\Theta}}{R} \right) \right] \equiv \epsilon_{R\Theta}^0 - Z \, k_{R\Theta}, \tag{10.24}$$

$$\epsilon_{RR} \equiv \varepsilon^3 \widehat{\epsilon}_{RR} = u_{,R} + \Gamma_{,R} w_{,R} - Z \, w_{,RR} \equiv \epsilon_{RR}^0 - Z \, k_{RR}, \tag{10.25}$$

$$\epsilon_{\Theta\Theta} \equiv \varepsilon^3 \widehat{\epsilon}_{\Theta\Theta} = \frac{v_{,\Theta} + u}{R} - Z \left(\frac{w_{,R}}{R} + \frac{w_{,\Theta\Theta}}{R^2} \right) \equiv \epsilon_{\Theta\Theta}^0 - Z k_{\Theta\Theta}, \tag{10.26}$$

where the Z-independent strains and "curvatures" are

$$\epsilon_{R\Theta}^0 \equiv \frac{1}{2} \left(v_{,R} + \frac{u_{,\Theta} - v}{R} + \frac{\Gamma_{,R} w_{,\Theta}}{R} \right), \qquad k_{R\Theta} \equiv -\frac{w_{,\Theta}}{R^2} + \frac{w_{,R\Theta}}{R}, \tag{10.27}$$

$$\epsilon_{RR}^0 \equiv u_{,R} + \Gamma_{,R} w_{,R}, \qquad k_{RR} \equiv w_{,RR}, \tag{10.28}$$

$$\epsilon_{\Theta\Theta}^0 \equiv \frac{v_{,\Theta} + u}{R}, \qquad k_{\Theta\Theta} \equiv \frac{w_{,R}}{R} + \frac{w_{,\Theta\Theta}}{R^2}. \tag{10.29}$$

Applying the scaling exponent values of this Section to the scaled equilibrium equations (7.38)–(7.40), we obtain to leading order:

$$\widehat{S}_{(0)RRi,\rho} + \frac{1}{\rho} \widehat{S}_{(0)R\Theta i,\Theta} + \widehat{S}_{(0)RZi,\varsigma} + \frac{1}{\rho} \left(\widehat{S}_{(0)RRi} - \widehat{S}_{(0)\Theta\Theta i} \right) = 0, \tag{10.30}$$

$$\widehat{S}_{(0)R\Theta i,\rho} + \frac{1}{\rho}\widehat{S}_{(0)\Theta\Theta i,\Theta} + \widehat{S}_{(0)\Theta Zi,\varsigma} + \frac{2}{\rho}\widehat{S}_{(0)R\Theta i} = 0, \tag{10.31}$$

$$\begin{aligned}\left(\widehat{\Gamma}_{,\rho}\widehat{S}_{(0)RRi} + \widehat{S}_{(0)RZi}\right)_{,\rho} &+ \frac{1}{\rho}\left(\widehat{\Gamma}_{,\rho}\widehat{S}_{(0)R\Theta i} + \widehat{S}_{(0)\Theta Zi}\right)_{,\Theta} \\ &+ \left(\widehat{\Gamma}_{,\rho}\widehat{S}_{(0)RZi} + \widehat{S}_{(0)ZZi}\right)_{,\varsigma} + \frac{1}{\rho}\left(\widehat{\Gamma}_{,\rho}\widehat{S}_{(0)RRi} + \widehat{S}_{(0)RZi}\right) + \widehat{g} = 0.\end{aligned} \tag{10.32}$$

In terms of the physical variables, these take the forms

$$S_{(0)RZi,Z} + \frac{1}{R}\left[\left(RS_{(0)RRi}\right)_{,R} - S_{(0)\Theta\Theta i} + S_{(0)R\Theta i,\Theta}\right] = 0, \tag{10.33}$$

$$S_{(0)\Theta Zi,Z} + \frac{1}{R^2}\left(R^2\, S_{(0)R\Theta i}\right)_{,R} + \frac{1}{R}S_{(0)\Theta\Theta i,\Theta} = 0. \tag{10.34}$$

$$\begin{aligned}\left[\Gamma_{,R}S_{(0)RZi} + S_{(0)ZZi}\right]_{,Z} &+ \frac{1}{R}\left[\Gamma_{,R}S_{(0)R\Theta i} + S_{(0)\Theta Zi}\right]_{,\Theta} \\ &+ \frac{1}{R}\left[R\left(\Gamma_{,R}S_{(0)RRi} + S_{(0)RZi}\right)\right]_{,R} + \rho_{0i}\, g = 0,\end{aligned} \tag{10.35}$$

The leading order boundary conditions of pressure are the same as for the previous theories, viz.,

$$S^{\pm}_{(0)RZ} = 0, \qquad S^{\pm}_{(0)\Theta Z} = 0, \qquad S^{\pm}_{(0)ZZ} = -p^{\pm}. \tag{10.36}$$

Integrating equations (10.33)–(10.35) through the thickness, we obtain the equilibrium equations by the same method used in Section 8 in terms of the stress resultants and couples introduced in equations (8.125)–(8.127), and (8.135)–(8.137), viz.:

$$\left(RN_R\right)_{,R} - N_\Theta + N_{R\Theta,\Theta} = 0, \tag{10.37}$$

$$\left(R^2 N_{R\Theta}\right)_{,R} + R\, N_{\Theta,\Theta} = 0, \tag{10.38}$$

and

$$\begin{aligned}&\left[R\,\Gamma_{,R}\, N_R + \left(R\, M_R\right)_{,R} - M_\Theta + M_{R\Theta,\Theta}\right]_{,R} \\ &\quad + \left[\Gamma_{,R}N_{R\Theta} + \frac{1}{R^2}\left(R^2\, M_{R\Theta}\right)_{,R} + \frac{1}{R}M_{\Theta,\Theta}\right]_{,\Theta} + \left(p + \gamma_0\, g\right)R = 0.\end{aligned} \tag{10.39}$$

The stress resultants and couples are again given by equations (8.141)–(8.143), i.e.,

$$N_R = \mathcal{N} + A\,\epsilon^0_{RR} + A_\nu\,\epsilon^0_{\Theta\Theta} + B\, k_{RR} + B_\nu\, k_{\Theta\Theta}, \tag{10.40}$$

$$N_{R\Theta} = A_\Theta\,\epsilon^0_{R\Theta} + B_\Theta\, k_{R\Theta}, \tag{10.41}$$

$$N_\Theta = \mathcal{N} + A_\nu\,\epsilon^0_{RR} + A\,\epsilon^0_{\Theta\Theta} + B_\nu\, k_{RR} + B\, k_{\Theta\Theta}, \tag{10.42}$$

$$M_R = -\mathcal{M} - B\,\epsilon^0_{RR} - B_\nu\,\epsilon^0_{\Theta\Theta} - D\, k_{RR} - D_\nu\, k_{\Theta\Theta}, \tag{10.43}$$

$$M_{R\Theta} = -B_\Theta \, \epsilon^0_{R\Theta} - D_\Theta \, k_{R\Theta}, \tag{10.44}$$

$$M_\Theta = -\mathcal{M} - B_\nu \, \epsilon^0_{RR} - B \, \epsilon^0_{\Theta\Theta} - D_\nu \, k_{RR} - D \, k_{\Theta\Theta}, \tag{10.45}$$

but note that the strains and curvatures are now replaced by (10.27)–(10.29) of the present Section.

11 Geometrically Linear Membrane Shell Laminate Theory

In this Section, we develop the equilibrium equations for a geometrically linear membrane laminate, subject to pressure and gravity loading. We begin by choosing the scaling exponents of Section 7 to be $r = m = 1$, $\ell = n = 3/2$, $t = 5/2$, $p = 3$, $q = 7/2$, and $\mu = 1/2$, as indicated in Table 1 at the end of that Section. With these exponents, equation (7.23) takes the form

$$\begin{aligned} \varepsilon^{7/2}\Sigma_i \widehat{S}_{ZZi} = \; & \varepsilon^{3/2}\Sigma_i \widehat{S}^{nm}_i + \mathcal{E}_i \Bigg\{ (1-\nu_i)\left[\widehat{W}_{,\varsigma} + \frac{1}{2}\widehat{W}^2_{,\varsigma} + \varepsilon \frac{1}{2}\left(\widehat{U}^2_{,\varsigma} + \widehat{V}^2_{,\varsigma} \right)\right] \\ & + \nu_i \Bigg[\varepsilon^{3/2} \left(\widehat{U}_{,\rho} + \frac{\widehat{V}_{,\Theta} + \widehat{U}}{\rho} \right) + \varepsilon^2 \,\widehat{\Gamma}_{,\rho} \widehat{W}_{,\rho} + \varepsilon^2 \left(\frac{1}{2}\widehat{W}^2_{,\rho} + \frac{\widehat{W}^2_{,\Theta}}{2\rho^2} \right) \\ & + \varepsilon^3 \left(\frac{\widehat{U}^2_{,\rho} + \widehat{V}^2_{,\rho}}{2} + \frac{(\widehat{U}_{,\Theta} - \widehat{V})^2 + (\widehat{V}_{,\Theta} + \widehat{U})^2}{2\rho^2} \right) \Bigg] \Bigg\}. \end{aligned} \tag{11.1}$$

The terms involving partial derivatives with respect to ς must be expanded up to terms of order $\varepsilon^{3/2}$, as in equations (9.2) and (9.3). Making the appropriate substitutions in (11.1), and then setting the coefficients of ε^0, $\varepsilon^{1/2}$, ε, and $\varepsilon^{3/2}$ in equation (11.1) to zero, we obtain the following four equations:

$$\widehat{W}_{(0),\varsigma} + \frac{1}{2}\widehat{W}^2_{(0),\varsigma} = 0, \tag{11.2}$$

$$\widehat{W}_{(1),\varsigma} + \widehat{W}_{(0),\varsigma}\, \widehat{W}_{(1),\varsigma} = 0, \tag{11.3}$$

$$\widehat{W}_{(2),\varsigma} + \frac{1}{2}(\widehat{W}^2_{(1),\varsigma} + 2\,\widehat{W}_{(0),\varsigma}\, \widehat{W}_{(2),\varsigma}) + \left(\widehat{U}^2_{(0),\varsigma} + \widehat{V}^2_{(0),\varsigma} \right) = 0, \tag{11.4}$$

$$\begin{aligned} \Sigma_i \widehat{S}^{nm}_i + \mathcal{E}_i \Bigg\{ & (1-\nu_i)\Big[\widehat{W}_{(3),\varsigma} + (\widehat{W}_{(0),\varsigma} \widehat{W}_{(3),\varsigma} + \widehat{W}_{(1),\varsigma}\, \widehat{W}_{(2),\varsigma}) \\ & + \left(\widehat{U}_{(0),\varsigma}\, \widehat{U}_{(1),\varsigma} + \widehat{V}_{(0),\varsigma}\, \widehat{V}_{(1),\varsigma} \right) \Big] + \nu_i \left(\widehat{U}_{(0),\rho} + \frac{\widehat{V}_{(0),\Theta} + \widehat{U}_{(0)}}{\rho} \right) \Bigg\} = 0. \end{aligned} \tag{11.5}$$

Equation (11.2) implies that $\widehat{W}_{(0),\varsigma} = 0$, hence $\widehat{W}_{(0)} = \widehat{w}(\rho, \zeta)$, and (11.3) then yields $\widehat{W}_{(1),\varsigma} = 0$, reducing the remaining two equations to

$$\widehat{W}_{(2),\varsigma} + \left(\widehat{U}^2_{(0),\varsigma} + \widehat{V}^2_{(0),\varsigma} \right) = 0, \tag{11.6}$$

$$\Sigma_i \widehat{S}^{nm}_i + \mathcal{E}_i \Bigg\{ (1-\nu_i)\Big[\widehat{W}_{(3),\varsigma} + \left(\widehat{U}_{(0),\varsigma}\, \widehat{U}_{(1),\varsigma} + \widehat{V}_{(0),\varsigma}\, \widehat{V}_{(1),\varsigma} \right) \Big] + \nu_i \left(\widehat{U}_{(0),\rho} + \frac{\widehat{V}_{(0),\Theta} + \widehat{U}_{(0)}}{\rho} \right) \Bigg\} = 0. \tag{11.7}$$

We turn next to the two off-diagonal constitutive relations (7.24) and (7.25) under these new scalings:

$$\varepsilon^3 \Sigma_i \widehat{S}_{RZi} = \frac{G_i}{2} \left[\varepsilon^{1/2}\, \widehat{U}_{,\varsigma} + \varepsilon \widehat{W}_{,\rho} + \varepsilon \widehat{\Gamma}_{,\rho} \widehat{W}_{,\varsigma} + \varepsilon \widehat{W}_{,\rho} \widehat{W}_{,\varsigma} + \varepsilon^2 \left(\widehat{U}_{,\rho} \widehat{U}_{,\varsigma} + \widehat{V}_{,\rho} \widehat{V}_{,\varsigma} \right) \right], \tag{11.8}$$

$$\varepsilon^3\Sigma_i\widehat{S}_{\Theta Zi} = \frac{G_i}{2}\left\{\varepsilon^{1/2}\widehat{V}_{,\zeta} + \varepsilon\frac{\widehat{W}_{,\Theta}}{\rho} + \varepsilon\frac{\widehat{W}_{,\zeta}\widehat{W}_{,\Theta}}{\rho} + \varepsilon^2\left[\frac{\widehat{U}_{,\zeta}(\widehat{U}_{,\Theta}-\widehat{V}) + \widehat{V}_{,\zeta}(\widehat{V}_{,\Theta}+\widehat{U})}{\rho}\right]\right\}. \tag{11.9}$$

The leading order terms in these equations imply $\widehat{U}_{(0),\zeta} = 0$ and $\widehat{V}_{(0),\zeta} = 0$, hence the leading order in-plane displacement components are both independent of ζ, i.e., $\widehat{U}_{(0)} = \widehat{u}(\rho,\Theta)$ and $\widehat{V}_{(0)} = \widehat{u}(\rho,\Theta)$. When these two results are substituted in (11.6) and (11.7) we find that $\widehat{W}_{(2),\zeta} = 0$, and $\widehat{W}_{(3),\zeta}$ is given by

$$\widehat{W}_{(3),\zeta} = -\frac{\Sigma_i\widehat{S}_i^{nm}}{\mathcal{E}_i(1-\nu_i)} - \frac{\nu_i}{(1-\nu_i)}\left(\widehat{U}_{(0),\rho} + \frac{\widehat{V}_{(0),\Theta}+\widehat{U}_{(0)}}{\rho}\right), \tag{11.10}$$

which will be used later. First, we note that the off-diagonal in-plane constitutive relation (7.26) has the following form under these scalings:

$$\begin{aligned}\varepsilon^{3/2}\Sigma_i\widehat{S}_{R\Theta i} = \frac{G_i}{2}\Bigg\{&\varepsilon^{3/2}\left(\widehat{V}_{,\rho} + \frac{\widehat{U}_{,\Theta}-\widehat{V}}{\rho}\right) + \varepsilon^2\frac{\widehat{\Gamma}_{,\rho}\widehat{W}_{,\Theta}}{\rho} + \varepsilon^2\frac{\widehat{W}_{,\rho}\widehat{W}_{,\Theta}}{\rho}\\ &+ \varepsilon^3\left[\frac{\widehat{U}_{,\rho}(\widehat{U}_{,\Theta}-\widehat{V}) + \widehat{V}_{,\rho}(\widehat{V}_{,\Theta}+\widehat{U})}{\rho}\right]\Bigg\}.\end{aligned} \tag{11.11}$$

The leading order term here yields the same constitutive relation found in (8.18), except that the strain-displacement relation has changed:

$$\Sigma_i\widehat{S}_{(0)R\Theta i} = \frac{G_i}{2}\left(\widehat{V}_{,\rho} + \frac{\widehat{U}_{,\Theta}-\widehat{V}}{\rho}\right) \equiv G_i\,\widehat{\epsilon}_{R\Theta}, \qquad \widehat{\epsilon}_{R\Theta} \equiv \frac{1}{2}\left(\widehat{V}_{,\rho} + \frac{\widehat{U}_{,\Theta}-\widehat{V}}{\rho}\right). \tag{11.12}$$

The final two constitutive relations (7.27) and (7.28) take the following forms:

$$\begin{aligned}\varepsilon^{3/2}\Sigma_i\widehat{S}_{RRi} = \varepsilon^{3/2}\Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\Bigg((1-\nu_i)&\left\{\varepsilon^{3/2}\widehat{U}_{,\rho} + \varepsilon^2\widehat{\Gamma}_{,\rho}\widehat{W}_{,\rho} + \frac{1}{2}\left[\varepsilon^2\widehat{W}_{,\rho}^2 + \varepsilon^3\left(\widehat{U}_{,\rho}^2 + \widehat{V}_{,\rho}^2\right)\right]\right\}\\ &+ \nu_i\left\{\varepsilon^{3/2}\left(\frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho}\right) + \varepsilon^2\frac{\widehat{W}_{,\Theta}^2}{2\rho^2} + \varepsilon^3\left[\frac{(\widehat{U}_{,\Theta}-\widehat{V})^2 + (\widehat{V}_{,\Theta}+\widehat{U})^2}{2\rho^2}\right]\right.\\ &\left.+ \widehat{W}_{,\zeta} + \frac{1}{2}\widehat{W}_{,\zeta}^2 + \varepsilon\frac{1}{2}\left(\widehat{U}_{,\zeta}^2 + \widehat{V}_{,\zeta}^2\right)\right\}\Bigg),\end{aligned} \tag{11.13}$$

and

$$\begin{aligned}\varepsilon^{3/2}\Sigma_i\widehat{S}_{\Theta\Theta i} = \varepsilon^{3/2}\Sigma_i\widehat{S}_i^{nm} + \mathcal{E}_i\Bigg((1-\nu_i)&\left\{\varepsilon^{3/2}\left(\frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho}\right) + \varepsilon^2\frac{\widehat{W}_{,\Theta}^2}{2\rho^2} + \varepsilon^3\left[\frac{(\widehat{U}_{,\Theta}-\widehat{V})^2 + (\widehat{V}_{,\Theta}+\widehat{U})^2}{2\rho^2}\right]\right\}\\ &+ \nu_i\left\{\varepsilon^{3/2}\widehat{U}_{,\rho} + \varepsilon^2\widehat{\Gamma}_{,\rho}\widehat{W}_{,\rho} + \frac{1}{2}\left[\varepsilon^2\widehat{W}_{,\rho}^2 + \varepsilon^3\left(\widehat{U}_{,\rho}^2 + \widehat{V}_{,\rho}^2\right)\right]\right.\\ &\left.+ \widehat{W}_{,\zeta} + \frac{1}{2}\widehat{W}_{,\zeta}^2 + \varepsilon\frac{1}{2}\left(\widehat{U}_{,\zeta}^2 + \widehat{V}_{,\zeta}^2\right)\right\}\Bigg).\end{aligned} \tag{11.14}$$

The terms involving derivatives with respect to ζ in the last two equations reduce to simply $\widehat{W}_{(3),\zeta}$, which can be replaced by (11.10). The coefficients of $\varepsilon^{3/2}$ then yield, after some algebra, the same in-plane constitutive relations (8.27) and (8.28), except that the strains are now given by the simpler expressions

$$\widehat{\epsilon}_{RR} \equiv \widehat{U}_{,\rho}, \qquad \widehat{\epsilon}_{\Theta\Theta} \equiv \frac{\widehat{V}_{,\Theta}+\widehat{U}}{\rho}. \tag{11.15}$$

As with the geometrically nonlinear membrane theory, we again find that the leading order in-plane displacement components $U_{(0)R} = u(R,\Theta)$ and $U_{(0)\Theta} = v(R,\Theta)$ are independent of Z. The curvature terms k_{RR}, $k_{R\Theta}$, and $k_{\Theta\Theta}$ are thus again absent in equations (8.35)–(8.37) defining the in-plane strain components. In addition, the nonlinear terms involving derivatives of w, as well as any reference to the surface-defining function $\Gamma(R)$, have disappeared from the strain-displacement relations. In terms of physical variables, the governing equations can be summarized thus far by:

$$U_{(0)Z} = w(R,\Theta), \qquad U_{(0)R} = u(R,\Theta), \qquad U_{(0)\Theta} = v(R,\Theta), \tag{11.16}$$

$$S_{(0)R\Theta i} = G_i\, \epsilon_{R\Theta}, \tag{11.17}$$

$$S_{(0)RRi} = S_i + \frac{E_i}{1-\nu_i^2}\left(\epsilon_{RR} + \nu_i \epsilon_{\Theta\Theta}\right), \tag{11.18}$$

$$S_{(0)\Theta\Theta i} = S_i + \frac{E_i}{1-\nu_i^2}\left(\epsilon_{\Theta\Theta} + \nu_i \epsilon_{RR}\right), \tag{11.19}$$

where

$$\epsilon_{R\Theta} = \frac{1}{2}\left(v_{,R} + \frac{u_{,\Theta} - v}{R}\right) \equiv \epsilon^0_{R\Theta}, \tag{11.20}$$

$$\epsilon_{RR} = u_{,R} \equiv \epsilon^0_{RR}, \tag{11.21}$$

$$\epsilon_{\Theta\Theta} = \frac{v_{,\Theta} + u}{R} \equiv \epsilon^0_{\Theta\Theta}. \tag{11.22}$$

Under these new scalings, the radial and circumferential equilibrium equations (8.122) and (8.123) are unchanged, but the out-of-plane shear stresses disappear from the axial equilibrium equation (8.124). The leading order equilibrium equations are then given by

$$S_{(0)RZi,Z} + \frac{1}{R}\left[\left(RS_{(0)RRi}\right)_{,R} - S_{(0)\Theta\Theta i} + S_{(0)R\Theta i,\Theta}\right] = 0, \tag{11.23}$$

$$S_{(0)\Theta Zi,Z} + \frac{1}{R^2}\left(R^2 S_{(0)R\Theta i}\right)_{,R} + \frac{1}{R} S_{(0)\Theta\Theta i,\Theta} = 0, \tag{11.24}$$

$$S_{(0)ZZi,Z} + \frac{1}{R}\left[\omega_{,R} S_{(0)R\Theta i} + \frac{w_{,\Theta}}{R} S_{(0)\Theta\Theta i}\right]_{,\Theta} + \frac{1}{R}\left[R\left(\omega_{,R} S_{(0)RRi} + \frac{w_{,\Theta}}{R} S_{(0)R\Theta i}\right)\right]_{,R} + \rho_{0i}\, g = 0. \tag{11.25}$$

These may now be integrated through the thickness, applying the boundary conditions of pressure, to obtain (after multiplying thru by R) equations in terms of the in-plane stress resultants (8.125)–(8.127):

$$\left(RN_R\right)_{,R} - N_\Theta + N_{R\Theta,\Theta} = 0, \tag{11.26}$$

$$\left(R^2 N_{R\Theta}\right)_{,R} + R N_{\Theta,\Theta} = 0, \tag{11.27}$$

$$\left[R\left(\omega_{,R} N_R + \frac{w_{,\Theta}}{R} N_{R\Theta}\right)\right]_{,R} + \left(\omega_{,R} N_{R\Theta} + \frac{w_{,\Theta}}{R} N_\Theta\right)_{,\Theta} + \left(p + \gamma_0\, g\right) R = 0. \tag{11.28}$$

Since the curvatures have vanished from the formulation, the stress resultants (8.141)–(8.143) in this case reduce to

$$N_R = \mathcal{N} + A\epsilon_{RR}^0 + A_\nu\,\epsilon_{\Theta\Theta}^0, \tag{11.29}$$

$$N_{R\Theta} = A_\Theta\,\epsilon_{R\Theta}^0, \tag{11.30}$$

$$N_\Theta = \mathcal{N} + A_\nu\,\epsilon_{RR}^0 + A\,\epsilon_{\Theta\Theta}^0. \tag{11.31}$$

12 Conclusions

The method of asymptotic expansions has been applied in a purely formal way to the geometrically nonlinear, three-dimensional equilibrium equations of a coated membrane laminate, each component of which is assumed to be a uniform, homogeneous, isotropic elastic material. Four different choices of the scaling exponents appearing in equations (7.2)–(7.5) have led to four distinct theories, each of which reduces to a well-known theory when specialized to a single material. The method is systematic and self-consistent, as the only freedom available in constructing a theory is the choice of scaling exponents. In short, having made a choice, one is led without benefit of any further assumptions to a theory dictated by that choice. The main weakness in the method is the essentially *ad hoc* nature of the choice of exponents used in the scaling of variables and loads. Attempts have been made in the recent literature (see, for example [17, 18]) to remedy this remaining unsatisfactory feature.

Although no attempt has been made here to put the method on a rigorous mathematical foundation, articles doing so are available in the literature, especially from the French school led by Ciarlet [19, 20, and references therein] and his co-workers. Our goal was to survey, and systematically derive by a single method, generalizations of several well-known theories of a single material to a coated membrane laminate. The mechanical behavior of such a laminate is of considerable practical interest to those in the aerospace community committed to the manufacture and deployment of large aperture, optical quality reflectors in space. The availability in a single publication of several models from which to choose for the analysis of such a laminate will hopefully be useful to other workers in the community. In that regard, a companion Volume II of this Report providing details of the analytical solutions of several boundary value problems associated with these models is currently being prepared for publication.

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A Generalization to a Multilayer Coating

High-reflectance optical coatings typically consist of a dielectric "stack" of two or more coating materials alternating in position through the thickness. In this Appendix we generalize our work to include such multilayer coatings. The strain-displacement relations and equilibrium equations characterizing a given theory are, except for the gravitational term of the axial equilibrium equation, unchanged by this generalization. The only calculations that require modification are the through-the-thickness integrals of the constitutive relations, which define the stress resultants and couples. These were first introduced in Subsection 8.6. We give details of the calculations for N_R and M_R, noting that those for N_Θ, $N_{R\Theta}$, M_Θ, and $M_{R\Theta}$ involve nothing new, and will be seen to be obvious generalizations of the earlier single-coating (two-layer) results.

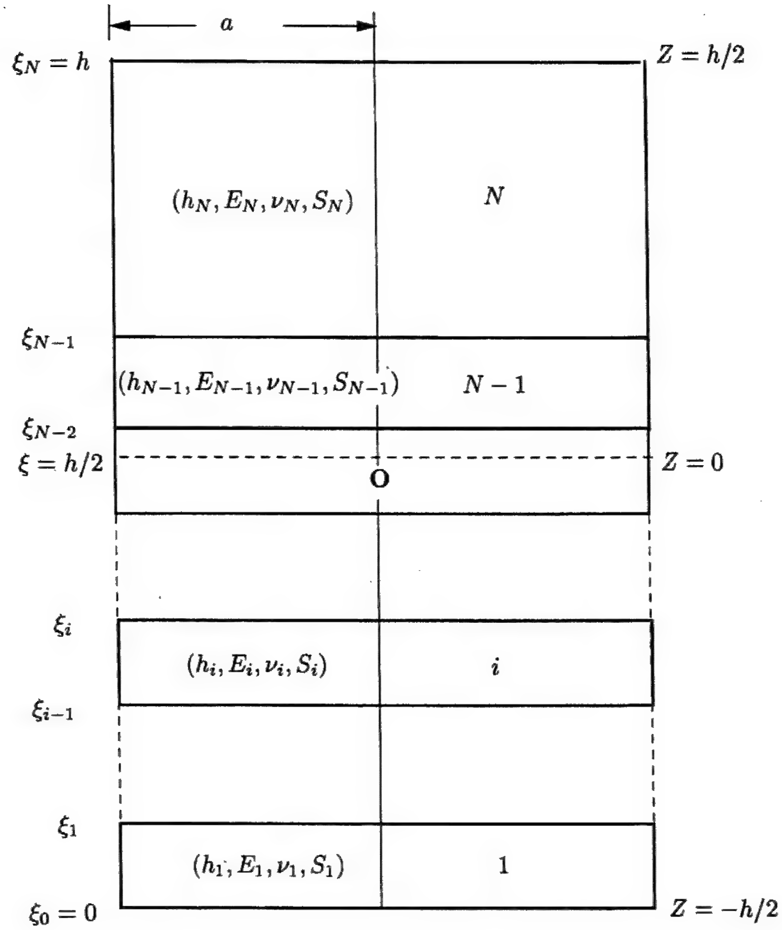

Figure 5: Geometry of multilayer stack.

The geometry of the reference placement of our membrane substrate layer N, with $N-1$ coatings, is illustrated in Figure 5 (compare to Figure 2). To do the through-the-thickness integrals, we introduce a change of variable to

$$\xi = Z + \frac{h}{2}. \tag{A.1}$$

As in Figure 2, the middle plane of the stack is the plane $Z = 0$ (corresponding to $\xi = h/2$) which has, for illustrative purposes only, been placed in the coating numbered $N-2$ in Figure 5. From this Figure, it can be seen that the thickness h_i of layer i is given in terms of the new coordinate ξ by

$$h_i = \xi_i - \xi_{i-1}, \tag{A.2}$$

and the total thickness is

$$h = \sum_{i=1}^{N} h_i = \sum_{i=1}^{N} (\xi_i - \xi_{i-1}) = \xi_N. \tag{A.3}$$

Thus, for the through-the-thickness integral of an arbitrary function $f(Z)$ we have:

$$\int_{Z=-h/2}^{h/2} f(Z)dZ = \int_{\xi=0}^{h} \widetilde{f}(\xi)d\xi = \sum_{i=1}^{N} \int_{\xi=\xi_{i-1}}^{\xi_i} \widetilde{f}(\xi)d\xi, \tag{A.4}$$

where $\widetilde{f}(\xi) \equiv f(\xi - h/2)$.

We begin by making this change of variables in each of the definitions (8.125)–(8.127) and (8.135)–(8.137) of the stress resultants and stress couples, respectively:

$$N_R \equiv \int_{-h/2}^{h/2} S_{(0)RRi}(Z)\,dZ = \sum_{i=1}^{N} \int_{\xi=\xi_{i-1}}^{\xi_i} \widetilde{S}_{(0)RRi}(\xi)d\xi, \tag{A.5}$$

$$N_{R\Theta} \equiv \int_{-h/2}^{h/2} S_{(0)R\Theta i}(Z)\,dZ = \sum_{i=1}^{N} \int_{\xi=\xi_{i-1}}^{\xi_i} \widetilde{S}_{(0)R\Theta i}(\xi)d\xi, \tag{A.6}$$

$$N_{\Theta} \equiv \int_{-h/2}^{h/2} S_{(0)\Theta\Theta i}(Z)\,dZ = \sum_{i=1}^{N} \int_{\xi=\xi_{i-1}}^{\xi_i} \widetilde{S}_{(0)\Theta\Theta i}(\xi)d\xi, \tag{A.7}$$

$$M_R \equiv \int_{-h/2}^{h/2} Z\,S_{(0)RRi}(Z)\,dZ = \sum_{i=1}^{N} \int_{\xi=\xi_{i-1}}^{\xi_i} (\xi - h/2)\,\widetilde{S}_{(0)RRi}(\xi)d\xi, \tag{A.8}$$

$$M_{R\Theta} \equiv \int_{-h/2}^{h/2} Z\,S_{(0)R\Theta i}(Z)\,dZ = \sum_{i=1}^{N} \int_{\xi=\xi_{i-1}}^{\xi_i} (\xi - h/2)\,\widetilde{S}_{(0)R\Theta i}(\xi)d\xi, \tag{A.9}$$

$$M_{\Theta} \equiv \int_{-h/2}^{h/2} Z\,S_{(0)\Theta\Theta i}(Z)\,dZ = \sum_{i=1}^{N} \int_{\xi=\xi_{i-1}}^{\xi_i} (\xi - h/2)\,\widetilde{S}_{(0)\Theta\Theta i}(\xi)d\xi. \tag{A.10}$$

Recall now the separation of each of the in-plane constitutive relations (8.32)-(8.34) into Z-dependent and Z-independent parts:

$$S_{(0)R\Theta i} = \sigma_{R\Theta i} - Z\eta_{R\Theta i}, \qquad S_{(0)RRi} = \sigma_{Ri} - Z\eta_{Ri}, \qquad S_{(0)\Theta\Theta i} = \sigma_{\Theta i} - Z\eta_{\Theta i}, \tag{A.11}$$

where the Z-independent functions are defined by

$$\sigma_{R\Theta i} \equiv G_i\,\epsilon^0_{R\Theta}, \qquad \eta_{R\Theta i} \equiv G_i\,k_{R\Theta}, \tag{A.12}$$

$$\sigma_{Ri} \equiv S_i + Q_i\left(\epsilon^0_{RR} + \nu_i\epsilon^0_{\Theta\Theta}\right), \qquad \eta_{Ri} \equiv Q_i\left(k_{RR} + \nu_i k_{\Theta\Theta}\right), \tag{A.13}$$

$$\sigma_{\Theta i} \equiv S_i + Q_i\left(\epsilon^0_{\Theta\Theta} + \nu_i\epsilon^0_{RR}\right), \qquad \eta_{\Theta i} \equiv Q_i\left(k_{\Theta\Theta} + \nu_i k_{RR}\right). \tag{A.14}$$

After the change of variables we have

$$\widetilde{S}_{(0)RRi} = \sigma_{Ri} - (\xi - h/2)\,\eta_{Ri}, \qquad \widetilde{S}_{(0)\Theta\Theta i} = \sigma_{\Theta i} - (\xi - h/2)\,\eta_{\Theta i}, \tag{A.15}$$

and

$$\widetilde{S}_{(0)R\Theta i} = \sigma_{R\Theta i} - (\xi - h/2)\,\eta_{R\Theta i}. \tag{A.16}$$

Substitution of the first of equations (A.15) in equation (A.5) yields

$$\begin{aligned} N_R &= \sum_{i=1}^{N} \int_{\xi=\xi_{i-1}}^{\xi_i} \left[\sigma_{Ri} - (\xi - h/2)\,\eta_{Ri}\right] d\xi, \\ &= \sum_{i=1}^{N} \left[\sigma_{Ri}\,(\xi_i - \xi_{i-1}) + \frac{h}{2}\eta_{Ri}\,(\xi_i - \xi_{i-1}) - \eta_{Ri}\frac{1}{2}\left(\xi_i^2 - \xi_{i-1}^2\right)\right], \\ &= \sum_{i=1}^{N} \left\{\sigma_{Ri}h_i + \frac{1}{2}\eta_{Ri}h_i\left[h - (\xi_i + \xi_{i-1})\right]\right\}, \end{aligned} \tag{A.17}$$

where we made use of (A.2) to obtain the final equality. Now, from (A.2) it is easy to show that

$$\xi_i = \sum_{k=1}^{i} h_k, \tag{A.18}$$

from which

$$\xi_i + \xi_{i-1} = \sum_{k=1}^{i} h_k + \sum_{k=1}^{i-1} h_k, \tag{A.19}$$

hence

$$h - (\xi_i + \xi_{i-1}) = h - \sum_{k=1}^{i} h_k - \sum_{k=1}^{i-1} h_k = \sum_{k=i}^{N} h_k - \sum_{k=1}^{i} h_k. \tag{A.20}$$

where the last result follows by substituting the first equality of (A.3) for h. Substitution of this result in (A.17) yields

$$N_R = \sum_{i=1}^{N} \sigma_{Ri}h_i + \frac{1}{2}\sum_{i=1}^{N} \eta_{Ri}h_i \left(\sum_{k=i}^{N} h_k - \sum_{k=1}^{i} h_k\right). \tag{A.21}$$

In this expression and others that follow we make use of the following identity, which can be proven using the principle of mathematical induction:

$$\sum_{i=1}^{N} a_i h_i \left(\sum_{k=i}^{N} h_k - \sum_{k=1}^{i} h_k\right) = \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} h_i h_k \,(a_i - a_k)\,, \tag{A.22}$$

for any indexed function a_i. This allows us to rewrite (A.21) as

$$N_R = \sum_{i=1}^{N} \sigma_{Ri}h_i + \frac{1}{2}\sum_{i=1}^{N-1} \sum_{k=i+1}^{N} h_i h_k \,(\eta_{Ri} - \eta_{Rk})\,. \tag{A.23}$$

Substituting now for σ_{Ri} and η_{Ri} from equation (A.13), we find

$$\begin{aligned} N_R = &\sum_{i=1}^{N} h_i \left[S_i + Q_i \left(\epsilon_{RR}^0 + \nu_i \epsilon_{\Theta\Theta}^0\right)\right] + \\ &\frac{1}{2}\sum_{i=1}^{N-1} \sum_{k=i+1}^{N} h_i h_k \left[Q_i \left(k_{RR} + \nu_i k_{\Theta\Theta}\right) - Q_k \left(k_{RR} + \nu_k k_{\Theta\Theta}\right)\right], \end{aligned} \tag{A.24}$$

which can be written in the same form as equation (8.141), i.e.,

$$N_R = \mathcal{N} + A\,\epsilon^0_{RR} + A_\nu\,\epsilon^0_{\Theta\Theta} + B\,k_{RR} + B_\nu\,k_{\Theta\Theta}, \tag{A.25}$$

where the multilayer constants are given by

$$\mathcal{N} = \sum_{i=1}^{N} h_i S_i, \tag{A.26}$$

$$A = \sum_{i=1}^{N} h_i Q_i, \qquad A_\nu = \sum_{i=1}^{N} h_i Q_i \nu_i,, \tag{A.27}$$

$$B = \frac{1}{2}\sum_{i=1}^{N-1}\sum_{k=i+1}^{N} h_i h_k\,(Q_i - Q_k)\,, \qquad B_\nu = \frac{1}{2}\sum_{i=1}^{N-1}\sum_{k=i+1}^{N} h_i h_k\,(Q_i\nu_i - Q_k\nu_k)\,. \tag{A.28}$$

The multilayer expression for N_Θ follows directly from (A.25) and the observation that N_R and N_Θ differ ultimately only by an interchange of ϵ^0_{RR} and $\epsilon^0_{\Theta\Theta}$, and k_{RR} and $k_{\Theta\Theta}$, which yields

$$N_\Theta = \mathcal{N} + A\,\epsilon^0_{\Theta\Theta} + A_\nu\,\epsilon^0_{RR} + B\,k_{\Theta\Theta} + B_\nu\,k_{RR}. \tag{A.29}$$

The multilayer expression for $N_{R\Theta}$ is an obvious generalization of (8.142), i.e.,

$$N_{R\Theta} = A_\Theta\,\epsilon^0_{R\Theta} + B_\Theta\,k_{R\Theta}, \tag{A.30}$$

where the constants generalize those found in (8.98) and (8.99):

$$A_\Theta = \sum_{i=1}^{N} h_i G_i, \qquad B_\Theta = \frac{1}{2}\sum_{i=1}^{N-1}\sum_{k=i+1}^{N} h_i h_k\,(G_i - G_k)\,. \tag{A.31}$$

Turning now to the calculation of M_R for a multilayer, we have from (A.8):

$$M_R = \sum_{i=1}^{N}\int_{\xi=\xi_{i-1}}^{\xi_i} (\xi - h/2)\,\widetilde{S}_{(0)RRi}\,d\xi = \sum_{i=1}^{N}\int_{\xi=\xi_{i-1}}^{\xi_i} \xi\,\widetilde{S}_{(0)RRi}\,d\xi - \frac{h}{2}\sum_{i=1}^{N}\int_{\xi=\xi_{i-1}}^{\xi_i} \widetilde{S}_{(0)RRi}\,d\xi.$$

Substituting from the first of equations (A.15), and using the definition (A.5) in the last term, the last equation yields

$$\begin{aligned}
M_R &= \sum_{i=1}^{N}\int_{\xi=\xi_{i-1}}^{\xi_i} \xi\,[\sigma_{Ri} - (\xi - h/2)\,\eta_{Ri}]\,d\xi - \frac{h}{2}N_R, \\
&= \sum_{i=1}^{N}\int_{\xi=\xi_{i-1}}^{\xi_i} \xi\left(\sigma_{Ri} + \frac{h}{2}\eta_{Ri}\right)d\xi - \sum_{i=1}^{N}\int_{\xi=\xi_{i-1}}^{\xi_i} \eta_{Ri}\,\xi^2\,d\xi - \frac{h}{2}N_R, \\
&= \frac{1}{2}\sum_{i=1}^{N}\left(\sigma_{Ri} + \frac{h}{2}\eta_{Ri}\right)\left(\xi_i^2 - \xi_{i-1}^2\right) - \frac{1}{3}\sum_{i=1}^{N}\eta_{Ri}\,\left(\xi_i^3 - \xi_{i-1}^3\right) - \frac{h}{2}N_R,
\end{aligned}$$

or, factoring the squared and cubed difference terms, and using (A.2) in the results:

$$\begin{aligned}
M_R = \frac{1}{2}\sum_{i=1}^{N}\sigma_{Ri}\,h_i\,(\xi_i + \xi_{i-1}) + \frac{h}{4}\sum_{i=1}^{N}\eta_{Ri}\,h_i\,(\xi_i + \xi_{i-1}) \\
- \frac{1}{3}\sum_{i=1}^{N}\eta_{Ri}\,h_i\left(\xi_i^2 + \xi_i\xi_{i-1} + \xi_{i-1}^2\right) - \frac{h}{2}N_R.
\end{aligned} \tag{A.32}$$

Using the last equation of (A.17) to replace N_R in (A.32), and collecting terms involving σ_{Ri} and η_{Ri} in that result, we obtain

$$M_R = -\frac{1}{2}\sum_{i=1}^{N} \sigma_{Ri}\, h_i\, (h - \xi_i - \xi_{i-1})$$
$$- \frac{1}{12}\sum_{i=1}^{N} \eta_{Ri}\, h_i \left[3h^2 - 6h\,(\xi_i + \xi_{i-1}) + 4\,\left(\xi_i^2 + \xi_i\xi_{i-1} + \xi_{i-1}^2\right)\right],$$

or, using (A.20) and (A.22) in the sum involving σ_{Ri}:

$$M_R = -\frac{1}{2}\sum_{i=1}^{N-1}\sum_{k=i+1}^{N} h_i h_k\, (\sigma_{Ri} - \sigma_{Rk})$$
$$- \frac{1}{12}\sum_{i=1}^{N} \eta_{Ri}\, h_i \left[3h^2 - 6h\,(\xi_i + \xi_{i-1}) + 4\,\left(\xi_i^2 + \xi_i\xi_{i-1} + \xi_{i-1}^2\right)\right]. \quad \text{(A.33)}$$

Substituting for σ_{Ri} and η_{Ri} from (A.13), we find that (A.33) can be written in the same form as equation (8.144), i.e.,

$$M_R = -\mathcal{M} - B\,\epsilon_{RR}^0 - B_\nu\,\epsilon_{\Theta\Theta}^0 - D\,k_{RR} - D_\nu\,k_{\Theta\Theta}, \quad \text{(A.34)}$$

where the multilayer constants B and B_ν are defined in (A.28), $\mathcal{M}$ is given by

$$\mathcal{M} = \frac{1}{2}\sum_{i=1}^{N-1}\sum_{k=i+1}^{N} h_i\, h_k\, (S_i - S_k), \quad \text{(A.35)}$$

and D, D_ν have the forms

$$D = \frac{1}{12}\sum_{i=1}^{N} Q_i\, h_i \left[3h^2 - 6h\,(\xi_i + \xi_{i-1}) + 4\,\left(\xi_i^2 + \xi_i\xi_{i-1} + \xi_{i-1}^2\right)\right], \quad \text{(A.36)}$$

$$D_\nu = \frac{1}{12}\sum_{i=1}^{N} Q_i\, \nu_i\, h_i \left[3h^2 - 6h\,(\xi_i + \xi_{i-1}) + 4\,\left(\xi_i^2 + \xi_i\xi_{i-1} + \xi_{i-1}^2\right)\right]. \quad \text{(A.37)}$$

These two coefficients can be simplified somewhat using the following identities:

$$3h^2 - 6h\,(\xi_i + \xi_{i-1}) \equiv 3\left[h^2 - 2h\,(h_i + 2\xi_{i-1})\right] \equiv 3\left[(h - h_i)^2 - h_i^2 - 4h\xi_{i-1}\right],$$

$$\xi_i^2 + \xi_i\xi_{i-1} + \xi_{i-1}^2 \equiv (\xi_i - \xi_{i-1})^2 + 3\xi_i\,\xi_{i-1} \equiv h_i^2 + 3\xi_i\,\xi_{i-1},$$

where we made use of the fact that $\xi_i + \xi_{i-1} \equiv h_i + 2\xi_{i-1}$ in the first of these. Substituting these results in, for example, (A.36), yields after simplifying:

$$D = \frac{1}{12}\sum_{i=1}^{N} Q_i\, h_i \left[h_i^2 + 3\,(h - h_i)^2 - 12\xi_{i-1}\,(h - \xi_i)\right], \quad \text{(A.38)}$$

and we note that the third term in brackets vanishes for $i = 1$ (since $\xi_0 = 0$) and $i = N$ (since $\xi_N = h$). Similarly, D_ν may be written as

$$D_\nu = \frac{1}{12}\sum_{i=1}^{N} Q_i\, \nu_i\, h_i \left[h_i^2 + 3\,(h - h_i)^2 - 12\xi_{i-1}\,(h - \xi_i)\right]. \quad \text{(A.39)}$$

The multilayer form for M_Θ follows directly from (A.34) by simply interchanging ϵ^0_{RR} and $\epsilon^0_{\Theta\Theta}$, and k_{RR} and $k_{\Theta\Theta}$, which yields

$$M_\Theta = -\mathcal{M} - B\,\epsilon^0_{\Theta\Theta} - B_\nu\,\epsilon^0_{RR} - D\,k_{\Theta\Theta} - D_\nu\,k_{RR}, \tag{A.40}$$

while the multilayer expression for $M_{R\Theta}$ is a generalization of (8.145), i.e.,

$$M_{R\Theta} = -B_\Theta\,\epsilon^0_{R\Theta} - D_\Theta\,k_{R\Theta}, \tag{A.41}$$

where B_Θ is given in (A.31), and the new constant D_Θ generalizes (8.112) (it is in fact the same as (A.38) with Q_i replaced by G_i):

$$D_\Theta = \frac{1}{12}\sum_{i=1}^{N} G_i\,h_i\left[h_i^2 + 3\,(h - h_i)^2 - 12\xi_{i-1}\,(h - \xi_i)\right]. \tag{A.42}$$

It is straightforward to check that our earlier results for the stress resultants and couples (for a membrane with a single coating) follow immediately from those determined in this Appendix by taking $N = 2$ (in which case $i = 1 = c$ corresponds to the coating, while $i = 2 = s$ corresponds to the membrane substrate).

Finally, for a multilayer coating the areal density γ_0 defined by equation (8.103), which determines the gravitational body force appearing in the axial equilibrium equation of each theory, must be generalized to

$$\gamma_0 = \sum_{i=1}^{N} h_i\,\rho_{0i}, \tag{A.43}$$

where ρ_{0i} is the mass density of the material in layer i.

References

[1] D. K. Marker, J. M. Wilkes, R. A. Carreras, J. R. Rotgé, C. H. Jenkins, and J. T. Ash in *Gossamer Spacecraft: Membrane and Inflatable Structures Technology for Space Applications*, C. H. Jenkins, ed., vol. 191 of *AIAA Progress in Astronautics and Aeronautics*, ch. 4, Fundamentals of Membrane Optics, pp. 111–201, American Instititute of Aeronautics and Astronautics, 2001.

[2] Y. C. Fung, *Foundations of Solid Mechanics*, Prentice-Hall, Englewood Cliffs, N.J., 1965.

[3] A. C. Eringen, *Nonlinear Theory of Continuous Media*, McGraw Hill, New York, 1962.

[4] J. E. Marsden and T. J. R. Hughes, *Mathematical Foundations of Elasticity*, Dover, New York, 1994.

[5] J. Bonet and R. D. Wood, *Nonlinear Continuum Mechanics for Finite Element Analysis*, Cambridge University Press, New York, 1997.

[6] C. A. Truesdell and R. A. Toupin, *The Classical Field Theories*, vol. III of *Encyclopedia of Physics*, pp. 226–793. Springer-Verlag, Berlin, 1960.

[7] R. W. Ogden, *Non-Linear Elastic Deformations*, Dover, New York, 1997.

[8] S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York, second ed., 1951.

[9] H. A. Erbay and E. S. Suhubi, "An asymptotic theory of thin hyperelastic plates–I. General theory," *International Journal of Engineering Science* **29**(4), pp. 447–466, 1991.

[10] H. A. Erbay and E. S. Suhubi, "An asymptotic theory of thin hyperelastic plates–II. Incompressible solids and an application of the general theory," *International Journal of Engineering Science* **29**(4), pp. 467–480, 1991.

[11] H. A. Erbay, "On the asymptotic membrane theory of thin hyperelastic plates," *International Journal of Engineering Science* **35**(2), pp. 151–170, 1997.

[12] J.-Q. Tarn, "An asymptotic theory for nonlinear analysis of multilayered anisotropic plates," *Journal of the Mechanics and Physics of Solids* **45**(7), pp. 1105–1120, 1997.

[13] H. Hencky, "Über den spannungszustand in kreisrunden platten mit verschwindender biegungssteif- figkeit," *Zeitschrift für Mathematik und Physik* **63**, pp. 311–317, 1915.

[14] J. D. Campbell, "On the theory of initially tensioned circular membranes subjected to uniform pressure," *Quarterly Journal of Mechanics and Applied Mathematics* **IX**, pp. 84–93, 1956.

[15] R. M. Jones, *Mechanics of Composite Materials*, Scripta Book Company, Washington, D. C., first ed., 1975.

[16] W. H. Wittrick, "Stability of a bimetallic disk: Part I," *Quarterly Journal of Mechanics and Applied Mathematics* **VI**, pp. 15–26, 1953.

[17] O. Millet, A. Hamdouni, and A. Cimetiere, "Dimensional analysis and asymptotic expansions of the equilibrium equations in nonlinear elasticity–Part I: The membrane model," *Archives of Mechanics* **50**(6), pp. 953–973, 1998.

[18] O. Millet, A. Hamdouni, and A. Cimetiere, "Dimensional analysis and asymptotic expansions of the equilibrium equations in nonlinear elasticity–Part II: The 2D von Kármán model," *Archives of Mechanics* **50**(6), pp. 975–1001, 1998.

[19] P. G. Ciarlet, "A justification of the von Kármán equations," *Archive for Rational Mechanics and Analysis* **73**, pp. 349–389, 1980.

[20] P. G. Ciarlet and J. C. Paumier, "A justification of the Marguerre-von Kármán equations," *Computational Mechanics* **1**, pp. 177–202, 1986.

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